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This report describes progress on the referenced contract for the fiscal year 1981. This progress includes several results in minimax linear estimation and control as well as results for general minimax hierarchical games and minimax jamming strategies.

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October 20, 1981

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2. Summary of Progress During Reporting Period

The primary direction of progress on this project during the reporting period has been toward the design of robust linear estimation and control procedures for uncertain models. Generally speaking, a robust procedure is one which is insensitive (in terms of performance) to small deviations from an assumed model. One of the most successful approaches to robust design is a game theoretic one in which a procedure is sought to have the best worst-case performance over a relevant class of models neighboring the assumed (i.e., nominal) model. Thus, the primary design philosophies in this study have been minimax-mean-square-error estimation and minimax quadratic control. Related game theoretic approaches to hierarchical stochastic decision problems and to antijamming strategies have also been considered during the reporting period. A brief description of the results obtained during the reporting period is contained in the following paragraphs. More complete details of these results can be found in the publications listed at the end of this discussion, copies of which are attached as appendices to this report.

The problem of robust linear smoothing of a stationary random signal with uncertain spectrum observed in additive noise with uncertain spectrum is considered in [6]. Here, a general solution to this problem is given for spectral uncertainty classes of a general type based on

Choquet capacities. This type of model includes standard uncertainty models such as contaminated mixtures as well as several topological models of uncertainty. Further, an extensive numerical study [8] indicates that the worst-case performance of the proposed technique is generally much better than that resulting from designs which simply ignore the presence of uncertainty. More general problems of robust estimation of stationary signals (including smoothing, filtering and prediction) also have been considered in the case of discrete time [9]. Here, a general minimax result is given from which robust solutions to a variety of problems follow straightforwardly.

Also considered in this study are the problems of state estimation and control in linear stochastic systems with uncertain noise statistics. In [7], two aspects of minimax MSE state estimation are considered. These are: minimax state estimation for single-variable systems with uncertain process or observation noise spectra and minimax state estimation for multivariable systems with white noises of uncertain componentwise correlation. In each problem, a minimax theorem is proven indicating that the robust state estimator is the minimum MSE filter for a least-favorable model. Thus, for example, in the second problem the robust solution is the Kalman-Bucy filter for a least-favorable pair of process and state noise covariance matrices. The related problem of minimax estimation with nonhomogeneous Poisson observations processes with uncertain rate functions has also been considered [3]. Here, analogies with the continuous-observations case are exploited to obtain straightforward minimax solutions to this problem. Minimax linear-quadratic control within the second formulation mentioned above (i.e., white noises with uncertain componentwise correlation) is considered in [4,5]. Results similar to those for state estimation in [7] are found to hold for this problem. Also, an interesting result concerning the separability of estimation and control is observed in this problem. In particular, it is seen that the minimax controller design is independent of that of the state observer, whereas the reverse is

not necessarily true. Thus, for example, the state observer for minimax control is different from the minimax state estimator of [7] for the open-loop system. This is in contrast to the analogous problem without uncertainty in which estimation and control objectives separate.

As noted above, game theoretic analyses have also been applied to the problems of hierarchical stochastic decision making and anti-jamming. In particular, [1] considers the class of stochastic multiperson, multicriteria decision problems (defined on general Hilbert spaces) with quadratic objective functionals, static information structure, and with a hierarchical structure with regard to the order in which desicions are announced. Here, a set of conditions is obtained under which a unique equilibrium solution exists and can be determined as the limit of an infinite sequence. Further, [2] considers the problem of transmitting a sequence of independent and identically distributed Gaussian random variables through a memoryless Gaussian wiretap channel with an intelligent jammer. Under a minimax MSE criterion, the complete set of strategies for the jammer and transmitter is obtained for this problem within power constrainst on transmitter and jammer. A variety of solutions is possible depending on the relative power constraints of the players and the noise levels in the transmission and wiretap channels.

3. Publications Reporting Research Supported by ONR Contract N00014-81-K-0014

[1] T. Basar, "Hierarchical Equilibrium Solutions in Stochastic Decision Problems Defined on General Hilbert Spaces," submitted for publication to Information and Control.

[2] T. Basar, "The Gaussian Test Channel with an Intelligent Jammer," submitted for publication to IEEE Transactions on Information Theory.

[3] E.A. Geraniotis and H.V. Poor, "Minimax Filtering Problems for Observed Poisson Processes with Uncertain Rate Functions," Proceedings of the 20th IEEE Conference of Decision and Control, San Diego, California, December 16-18, 1981 (to appear).

[4] D.P. Looze, H.V. Poor, K.S. Vastola and J.C. Darragh, "Minimax Control of Linear Stochastic Systems with Noise Uncertainty," submitted to IEEE Transactions on Automatic Control.

[5] D.P. Looze, H.V. Poor, K.S. Vastola and J.C. Darragh, "Minimax Linear-Quadratic-Gaussian Control of Systems with Uncertain Statistics," Proceedings of the 10th IFIP Conference on System Modeling and Optimization. Springer-Verlag: New York, 1981 (to appear).

[6] H.V. Poor, "Minimax Linear Smoothing for Capacities," Annals of Probability (to appear).

[7] H.V. Poor and D.P. Looze, "Minimax State Estimation for Linear Stochastic Systems with Noise Uncertainty," IEEE Transactions on Automatic Control, pp. 902-906, August 1981.

[8] K.S. Vastola and H.V. Poor, "An Analysis of the Effects of Spectral Uncertainty on Wiener Filtering," submitted for publication to Automatica.

[9] K.S. Vastola and H.V. Poor, "Robust Linear Estimation of Stationary Discrete-Time Signals," Proceedings of the 1981 Conference on Information Sciences and Systems, The Johns Hopkins University, Baltimore, MD, March, 1981, pp. 512-516.

4. Appendix: Copies of Publications

This appendix contains copies of the publications listed in Section 3 above, with the exception of [5] which is currently being put into final form.

[1] T. Basar, "Hierarchical Equilibrium Solutions in Stochastic Decision Problems Defined on General Hilbert Spaces," submitted for publication to Information and Control.

HIERARCHICAL EQUILIBRIUM SOLUTIONS IN STOCHASTIC DECISION PROBLEMS DEFINED ON GENERAL HILBERT SPACES*

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ABSTRACT:

This paper considers the class of stochastic multi-person multi-criteria decision problems defined on general Hilbert spaces, with quadratic objective functionals, static information structure, and with the mode of decision making requiring the decision makers to announce their strategies in a sequential order. A set of conditions, independent of the probabilistic structure of the problem, is obtained, under which the hierarchical equilibrium solution exists, is unique and can be determined as the convergent limit of an infinite sequence. The analysis is confined primarily to the three-person case, in which context explicit conditions and strategies are obtained, but extensions to the case of more than three decision makers are also elucidated.

*Manuscript prepared June 10, 1981.

1. INTRODUCTION

This paper introduces and discusses a general approach towards derivation of the optimum (hierarchical equilibrium) solution of a class of stochastic multi-person multi-criteria decision problems which incorporate multi levels of hierarchy in decision making. Analysis is primarily confined to three-person decision problems defined on general inner-product spaces, with quadratic objective functionals, and with the mode of decision making requiring the decision makers to announce their policies in a sequential order; but, extensions to other types of decision problems with more than three decision makers and other modes of decision making are also discussed. The stochastic decision problems covered by our general framework are those with finite dimensional control (decision) spaces, those defined in continuous-time, with lumped or distributed parameters, as well as the ones whose state dynamics are described by differential-delay equations.

One of the important results obtained in the paper is that under suitable conditions (which are explicitly obtained), independent of the probabilistic structure of the problem, the equilibrium solution is unique and it can be determined as the convergent limit of an infinite sequence. For the special case of Gaussian distributions [such as the cases when all primitive random variables are Gaussian vectors (in the finite dimensional case), or are Gaussian stochastic processes (in the continuous-time case)] the optimum strategies of the decision makers are affine functions of the available static information.

Two special versions of this problem have been considered before for the two-person case. Başar (1980) discusses the case when the decision variables belong to finite dimensional spaces, and Bagchi and Başar (1981)

discuss the continuous-time version when the decision makers make noisy observations of the initial state. In a way, the present paper presents nontrivial extensions of these results to the $M(\geq 3)$-person case, but the solution here is not as explicit (in analytic form) as in those two papers because the framework here is more general.

A precise mathematical formulation of the problem is presented in Section 2, and the general solution is obtained in Section 3. For some background material on functional analysis that is employed in these two sections, the reader is referred to Balakrishnan (1976) and Kantorovich and Akhilov (1977). Section 4 treats a special case, and Section 5 discusses possible extensions to more general models. The paper concludes with an Appendix.

2. GENERAL FORMULATION

Let $(\Omega, \mathfrak{F}, \mathcal{P})$ be an underlying probability space, and H_i $(i = 0,1,2,3)$ be separable Hilbert spaces with inner products $(h^1, h^2)_i$, for $h^1, h^2 \in H_i$. Let z_i be an H_i-valued weak random variable [cf. Balakrishnan (1976)] defined on $(\Omega, \mathfrak{F}, \mathcal{P})$, and σ_i denote the sigma-algebra generated by z_i $(i = 1,2,3)$. Furthermore, z_i has a finite second moment, i.e.

(i) $E\{|(z_i, h_i)_i|^2\} < \infty$ for every $h_i \in H_i$

(ii) $E\{|(z_i, h_i)_i|^2\}$ is continuous in h_i,

where $E\{\cdot\}$ denotes the total expectation over the underlying probability space.

Let S_i $(i = 1,2,3)$ be separable Hilbert spaces with inner products $\langle s^1, s^2\rangle_i$, for $s^1, s^2 \in S_i$, and u_i be an S_i-valued weak random variable defined on $(\Omega, \mathfrak{F}, \mathcal{P})$ and satisfying the following two properties:

(i) u_i is σ_i-measurable,

(ii) u_i has finite second moments.

We call such a u_i a permissible <u>control</u> (<u>decision</u>) <u>variable</u> of the i-th decision maker [abbreviated, DMi]. Equivalently, we can introduce a permissible <u>strategy</u> for DMi as a mapping $\gamma_i : H_i \to S_i$ such that $\gamma_i(z_i)$ is σ_i-measurable and has finite second moments. Denote the class of all such mappings for DMi by Γ_i [to be called the <u>strategy</u> <u>space</u> of DMi], which is in fact a Hilbert space under the inner product $E\{\langle\gamma_i^1(z_i), \gamma_i^2(z_i)\rangle_i\}$, for $\gamma_i^1, \gamma_i^2 \in \Gamma_i$. In this general formulation [using the standard terminology of decision theory], z_o denotes the <u>state</u> of Nature and z_i $(i = 1,2,3)$ denotes the measurement available to DMi -- all these (weak) random variables are <u>static</u>, in the sense that they do not depend on the controls (actions) of the decision makers.

In order to complete the formulation, we introduce general quadratic objective (cost) functionals for the decision makers on the product strategy spaces $\Gamma_1 \times \Gamma_2 \times \Gamma_3$ as

$$J_i(\gamma_1, \gamma_2, \gamma_3) = E\{g_i(z_o, u_1, u_2, u_3) | u_j = \gamma_j(z_j),\ j = 1,2,3\} \quad (2.1)$$

where

$$\begin{aligned} g_i(z_o, u_1, u_2, u_3) &= \frac{1}{2} \langle u_i, u_i \rangle_i - \langle u_i, D_{ij} u_j \rangle_i - \langle u_i, D_{ik} u_k \rangle_i \\ &+ \frac{1}{2} \langle u_j, F^i_{jj} u_j \rangle_j + \frac{1}{2} \langle u_k, F^i_{kk} u_k \rangle_k - \langle u_j, F^i_{jk} u_k \rangle_j - \langle u_i, C^i_i z_o \rangle_i \quad (2.2) \\ &- \langle u_j, C^i_j z_o \rangle_j - \langle u_k, C^i_k z_o \rangle_k ; \quad i, j, k, = 1, 2, 3, \quad j \neq k \neq i,\ j < k. \end{aligned}$$

Here, D_{ij}, D_{ik}, F^i_{jj}, F^i_{kk}, F^i_{jk}, C^i_i, C^i_j, C^i_k are linear bounded operators defined on appropriate Hilbert spaces, with F^i_{jj} and F^i_{kk} being also self-adjoint.

Our objective in this paper is to investigate the existence, uniqueness and derivation of a hierarchical equilibrium solution for this stochastic decision problem, in the presence of a linear hierarchy for decision making. Specifically, it is assumed that the strategies are announced sequentially -- first DM1 announces his strategy and makes it known to both DM2 and DM3, then DM2 announces his strategy and makes it known to DM3, and finally DM3 decides on his optimal strategy. Each DM strives to obtain a minimum value for his cost functional, thereby leading to the following definition of a hierarchical equilibrium solution:

Definition 2.1. The set of strategies $\{\gamma_i^o \in \Gamma_i,\ i = 1, 2, 3\}$ is in hierarchical equilibrium if

(i) $J_1(\gamma_1^o, T_2(\gamma_1^o), T_3[\gamma_1^o, T_2(\gamma_1^o)]) \leq J_1(\gamma_1, T_2(\gamma_1), T_3[\gamma_1, T_2(\gamma_1)])$

$\forall \gamma_1 \in \Gamma_1$

(ii) $\gamma_2^o = T_2(\gamma_1^o)$

(iii) $\gamma_3^o = T_3(\gamma_1^o, \gamma_2^o)$

where T_2: $\Gamma_1 \to \Gamma_2$, T_3: $\Gamma_1 \times \Gamma_2 \to \Gamma_3$ are <u>unique</u> measurable mappings satisfying the inequalities $J_2(\gamma_1, T_2(\gamma_1), T_3[\gamma_1, T_2(\gamma_1)]) \leq J_2(\gamma_1, \gamma_2, T_3(\gamma_1,\gamma_2))$, $\forall\ \gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$, and $J_3(\gamma_1, \gamma_2, T_3(\gamma_1, \gamma_2)) \leq J_3(\gamma_1, \gamma_2, \gamma_3)$, $\forall\ \gamma_i \in \Gamma_i$, $i = 1, 2, 3$. □

<u>Remark 2.1</u>. The assumption of uniqueness of T_2 and T_3 in Def. 2.1 is not restrictive here since the underlying information structure is static, and such unique mappings exist, as we shall demonstrate later. If the information structure had been dynamic, however, we would have to extend the definition in order to account for nonunique responses [see Başar (1981a,b), for such an extension]. □

3. DERIVATION OF THE EQUILIBRIUM SOLUTION

Step 1. Firstly, to determine T_3, we hold $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ fixed and minimize $J_3(\gamma_1, \gamma_2, \cdot)$ over Γ_3. Since g_3 is strictly convex in u_3, this minimization problem admits the unique solution

$$\gamma_3(z_3) = T_3(\gamma_1,\gamma_2)(z_3) = E\{D_{31}\gamma_1(z_1) + D_{32}\gamma_2(z_2) + C_3^3 z_o | z_3\} \qquad (3.1)$$

where $E\{\cdot|z_3\}$ denotes the conditional expected value of $(\cdot)$ given the observed value of z_3. This is a well-defined quantity, which is σ_3-measurable, by standard results of probability theory [cf. Loève (1963)].

Step 2. Next, to determine T_2, we substitute (3.1) into $J_2(\gamma_1, \gamma_2, \gamma_3)$ and seek to minimize the resulting expression over $\gamma_2 \in \Gamma_2$ for fixed $\gamma_1 \in \Gamma_1$; this, however, is not a standard stochastic optimization problem, because of the presence of several conditional expectations. Writing out the function to be minimized, we have

$$\begin{aligned}
&J_2(\gamma_1, \gamma_2, T_3(\gamma_1, \gamma_2)) = E\{\tfrac{1}{2}< u_2, u_2>_2 - <u_2, D_{21}u_1>_2 \\
&- <u_2, D_{23}D_{31}E[u_1|z_3] + D_{23}D_{32}E[u_2|z_3] + D_{23}C_3^3E[z_o|z_3]>_2 \\
&+ \tfrac{1}{2}<u_1, F_{11}^2u_1>_1 + \tfrac{1}{2}<D_{31}E[u_1|z_3] + D_{32}E[u_2|z_3] + C_3^3 E[z_o|z_3], F_{33}^2D_{31}E[u_1|z_3] \\
&+ F_{33}^2D_{32}E[u_2|z_3] + F_{33}^2C_3^3E[z_o|z_3]>_3 - <u_1, F_{13}^2D_{31}E[u_1|z_3] + F_{13}^2D_{32}E[u_2|z_3] \\
&+ F_{13}^2C_3^3E[z_o|z_3]>_1 - <u_2, C_2^2z_o>_2 - <u_1, C_1^2z_o>_1 - <D_{31}E[u_1|z_3] \\
&+ D_{32}E[u_2|z_3] + C_3^3E[z_o|z_3], C_3^2z_o>_3 \;|u_i = \gamma_i(z_i),\ i = 1, 2\} \\
&\triangleq L_2(\gamma_2), \quad \text{for fixed } \gamma_1 \in \Gamma_1 .
\end{aligned}$$

We now seek a $\tilde{\gamma}_2 \in \Gamma_2$ so that, for fixed $\gamma_1 \in \Gamma_1$,

$$L_2(\tilde{\gamma}_2) < L_2(\gamma_2), \quad \forall\ \gamma_2 \in \Gamma_2. \tag{3.2}$$

Since Γ_2 is a linear vector space, to every $\gamma_2 \in \Gamma_2$ there corresponds an $h \in \Gamma_2$ such that $\gamma_2 = \tilde{\gamma}_2 + h$. Hence, (3.2) can be written as

$$L_2(\tilde{\gamma}_2) < L_2(\tilde{\gamma}_2 + h), \qquad \forall\ h \in \Gamma_2,$$

and furthermore, since L_2 is quadratic, we have

$$L_2(\tilde{\gamma}_2 + h) = L_2(\tilde{\gamma}_2) + \delta L_2(\tilde{\gamma}_2;h) + \frac{1}{2}\delta^2 L_2(\tilde{\gamma}_2;h),$$

whereby

$$L_2(\tilde{\gamma}_2) < L_2(\tilde{\gamma}_2) + \delta L_2(\tilde{\gamma}_2;h) + \frac{1}{2}\delta^2 L_2(\tilde{\gamma}_2;h), \quad \forall\ h \in \Gamma_2. \tag{3.3}$$

Here, δL_2 and $\delta^2 L_2$ denote, respectively, the first and second Gateaux variations of L_2 around $\tilde{\gamma}_2$, which are defined by

$$\delta L_2(\tilde{\gamma}_2;h) = \lim_{\alpha \to 0} \frac{d}{d\alpha} L_2(\tilde{\gamma}_2 + \alpha h)$$
$$\delta^2 L_2(\tilde{\gamma}_2;h) = \lim_{\alpha \to 0} \frac{d^2}{d\alpha^2} L_2(\tilde{\gamma}_2 + \alpha h).$$

The first of these is homogeneous of degree one, and therefore (3.3) readily leads to the set of necessary and sufficient conditions

$$\delta L_2(\tilde{\gamma}_2;h) = 0, \quad \delta^2 L_2(\tilde{\gamma}_2;h) > 0 \quad \forall\ h \in \Gamma_2. \tag{3.4}$$

By making use of the smoothing property of conditional expectations, we can write δL_2 and $\delta^2 L_2$ in the forms

$$\delta L_2(\tilde{\gamma}_2;h) = E\{\langle h(z_2), \ell_2(z_2,u_2,u_1)\rangle_2 \,|\, u_1 = \gamma_1(z_1),\ u_2 = \tilde{\gamma}_2(z_2)\} \tag{3.5a}$$

and

$$\delta^2 L_2(\tilde{\gamma}_2;h) = E\{\langle h(z_2),\ h(z_2)\rangle_2 + \langle E[h(z_2)\,|z_3],$$

$$(D^*_{32}F^2_{33}D_{32} - D_{23}D_{32} - D^*_{32}D^*_{23})\; E[h(z_2)|z_3]>_2\} \tag{3.5b}$$

where

$$\begin{aligned}\ell_2(z_2,u_2,u_1) &= u_2 + [D^*_{32}F^2_{33}D_{32} - D_{23}D_{32} - D^*_{32}D^*_{23}]\; E[E[u_2|z_3]|z_2] \\ &- D_{21}E[u_1|z_2] + [D^*_{32}F^2_{33}D_{31} - D_{23}D_{31} - D^*_{32}F^{2*}_{13}]E[E[u_1|z_3]|z_2] \\ &- [D_{23}C^3_3 + D^*_{32}C^2_3 - D^*_{32}F^2_{33}C^3_3]E[E[z_o|z_3]|z_2] - C^2_2E[z_o|z_2] ,\end{aligned}$$

and super-index star (*) denotes the adjoint operator under the appropriate inner product. Relations (3.4) and (3.5a) now readily lead to the first-order condition

$$\ell_2(z_2,\, u_2,\, u_1) \equiv 0 , \quad \text{a.e. } P_{z_2},\quad u_2 = \tilde{\gamma}_2(z_2) ,$$

where P_{z_2} is the probability measure induced under the weak random variable z_2. Let us rewrite this equality more explicitly as

$$\begin{aligned}\tilde{\gamma}_2(z_2) &= K\; E[E[\tilde{\gamma}_2(z_2)|z_3]|z_2] + D_{21}E[\gamma_1(z_1)|z_2] \\ &+ G_1E[E[\gamma_1(z_1)|z_3]|z_2] + C^2_2E[z_o|z_2] + G_2E[E[z_o|z_3]|z_2]\end{aligned} \tag{3.6}$$

where

$$K = -D^*_{32}F^2_{33}D_{32} + D_{23}D_{32} + D^*_{32}D^*_{23} \tag{3.7a}$$

$$G_1 = -D^*_{32}F^2_{33}D_{31} + D_{23}D_{31} + D^*_{32}F^{2*}_{13} \tag{3.7b}$$

$$G_2 = -D^*_{32}F^2_{33}C^3_3 + D_{23}C^3_3 + D^*_{32}C^2_3 \quad . \tag{3.7c}$$

Relation (3.6) is in fact a linear operator equation which has to be solved for $\tilde{\gamma}_2$, and this determines the mapping T_2 introduced in Def. 2.2. The questions of existence and uniqueness of such a mapping, and satisfaction

of the second condition of (3.4), are now addressed to in Proposition 3.1 to follow.

Preliminary terminology and notation

Let $\mathcal{B}_i$ be the space of linear bounded operators mapping Γ_i into itself. For $B \in \mathcal{B}_i$, let $\mu_i(B)$ and $||B||_i$ denote, respectively, the spectrum and norm of B, where the latter is defined by

$$||B||_i = \sup_{x \in \Gamma_i} [|\langle B_i x, B_i x\rangle_i|/\langle x, x\rangle_i]^{\frac{1}{2}} \tag{3.8}$$

Furthermore, let $r_i(B)$ denote the spectral radius of $B \in \mathcal{B}_i$, which is defined by

$$r_i(B) = \limsup_{n \to \infty} ||B^n||_i^{1/n} \tag{3.9}$$

Finally, let us introduce the Hilbert space M_i^j as the space of all S_i-valued σ_j-measurable weak random variables defined on $(\Omega, \mathcal{F}, P)$ and with finite second moments $(i, j = 1, 2, 3)$. [Note that M_i^i is in fact isomorphic to Γ_i, and it is also a Hilbert space.]

Now, by invoking the following condition

C1. $\quad r_2(K) < 1,$

we are in a position to state the proposition given below.

Proposition 3.1. Assume condition C1 to hold true. i) Equation (3.6) admits a unique solution $\tilde{\gamma}_2 \in \Gamma_2$ for every fixed $\gamma_1 \in \Gamma_1$, which is also the unique solution satisfying (3.4); hence $T_2: \Gamma_1 \to \Gamma_2$ is uniquely defined. ii) The unique solution to (3.6) can be found, for any fixed $\gamma_1 \in \Gamma_1$, as

the convergent limit of an iterative procedure (known as successive approximations), which starts with an arbitrary element of Γ_2 on the right-hand side (RHS) of (3.6) and recursively updates this choice by resubstituting (to the RHS) the strategy obtained on the left-hand side of (3.6).

Proof. Let us rewrite (3.6) as

$$\tilde{\gamma}_2(z_2) = K\,\mathcal{E}_2^2\mathcal{E}_2^3\tilde{\gamma}_2(z_2) + k_{\gamma_1}(z_2) \tag{3.10}$$

where k_{γ_1} is a fixed element of Γ_2 for every fixed $\gamma_1 \in \Gamma_1$, and $\mathcal{E}_2^3$ denotes the conditional expectation operator $\mathcal{E}_2^3\gamma_2(z_2) = E[\gamma_2(z_2)|z_3]$, where the convention is such that the super-index stands for the conditioning sigma-field and the sub-index identifies the range space $[S_1, S_2$ or $S_3]$ of the weak random variable whose conditional expectation is taken. The operator $\mathcal{E}_2^2$ is likewise defined as a conditional expectation operator. Each of these operators is a projection operator [see e.g. Başar (1975)], with $\mathcal{E}_2^i$ mapping M_2 into M_2^i, where M_2 is the Hilbert space of all second-order S_2-valued weak random variables defined on $(\Omega, \mathfrak{F}, \mathcal{P})$. [Note that $M_2 \supset M_2^1$ and $M_2 \supset M_2^2$.] Being projection operators, both $\mathcal{E}_2^2$ and $\mathcal{E}_2^3$ have unit norm, are linear and bounded. Therefore $K\mathcal{E}_2^2\mathcal{E}_2^3\colon M_2 \to M_2^2$ is a linear and bounded operator. We may also take the range spaces of $\mathcal{E}_2^2, \mathcal{E}_2^3$ and K as M_2 (instead of M_2^2) and introduce a natural extension of r_2 to the space of linear bounded operators mapping M_2 into itself, to be denoted $\bar{r}_2$. Then we have, using the spectral radius inequality for product operators,

$$\begin{aligned}\bar{r}_2(K\mathcal{E}_2^2\mathcal{E}_2^3) &\le \bar{r}_2(K)\bar{r}_2(\mathcal{E}_2^3)\bar{r}_2(\mathcal{E}_2^3)\\ &\le \bar{r}_2(K) = r_2(K) < 1,\end{aligned} \tag{i}$$

where the second inequality follows from a known (unit norm) property of

projection operators, the equality follows since (by construction) the restriction of $\bar{r}_2$ to M_2^2 is r_2, and the last inequality follows from C1.

Hence, the spectral radius of the operator in (3.10) is less than unity [or, equivalently, the spectrum of that operator is totally in the unit sphere], and by Theorem 3 (Chapter XIII) of Kantorovich and Akhilov (1977) equation (3.10) admits a unique solution in M_2. Furthermore, by the same theorem, the solution can be obtained iteratively by

$$\gamma_2^{(n+1)}(z_2) = K\delta_2^2\delta_2^3\gamma_2^{(n)}(z_2) + k_{\gamma_1}(z_2)\ , \quad n = 0, 1, \ldots$$

where $\gamma_2^{(0)}(\cdot)$ is any initial choice in M_2. Since we already know that the range space of $K\delta_2^2\delta_2^3$ comprises only σ_2-measurable elements, and $k_{\gamma_1} \in \Gamma_2$, it follows that $\tilde{\gamma}_2(\cdot) = \lim_{n\to\infty} \gamma_2^{(n)}(\cdot)$ is necessarily σ_2-measurable, and therefore the unique solution $\gamma_2(z_2)$ of (3.10) in M_2 is in fact in the subspace M_2^2, and to this, there corresponds a unique element $\tilde{\gamma}_2$ in Γ_2.

What remains to be shown now, in order to complete the proof of the Proposition, is satisfaction of the second order (sufficiency) condition of (3.4). Towards this end, we first note that K is a self-adjoint operator, and hence under C1,

$$|\langle x, Kx\rangle_1| < \langle x, x\rangle_1\ , \quad \forall\, x \in M_2^3\ , \quad x \neq \theta\ ,$$

where θ is the zero element in M_2^3. If this inequality is utilized in (3.5b), we readily arrive at the bound

$$\delta^2 L_2(\tilde{\gamma}_2;h) > E\{\langle h(z_2), h(z_2)\rangle_2 - \langle E[h(z_2)|z_3], E[h(z_2)|z_3]\rangle_2\}\ , \quad h \neq \theta.$$

By the nonexpansive property of conditional expectations [cf. Başar (1975)],

$$E\{\langle E[h(z_2)|z_3], E[h(z_2)|z_3]\rangle_2\} \leq E\{\langle h(z_2), h(z_2)\rangle_2\}\ ,$$

and therefore

$$\delta^2 L_2(\tilde{\gamma}_2;h) > E\{<E[h(z_2)|z_3], E[h(z_2)|z_3]>_2 - E[h(z_2)|z_3], E[h(z_2)|z_1]>_2\} = 0,$$

that is, the second Gateaux variation is positive definite. □

Remark 3.1. Under C1, the operator $[I-K\&_2^2\&_2^3]$ is invertible, and its inverse is also linear and bounded. This leads to an operator-form characterization of T_2, which is

$$T_2(\gamma_1)(z_2) = [I - K\&_2^2\&_2^3]^{-1}\{(D_{21}\&_1^2 + G_1\&_1^2\&_1^3)\gamma_1(z_1) + \&_2^2C_2^2Z_o + \&_2^2\&_2^3G_2z_o\} \tag{3.11}$$

where the definition of $\&_1^1$ is analgous to that of $\&_2^1$ in the proof of Proposition 3.1. We furthermore note that, under C1 the inverse can be written in the form of an infinite convergent series [cf. Kantorovich and Akhilov (1977)]

$$[I - K\&_2^2\&_2^3]^{-1} = I + \sum_{n=1}^{\infty}(K\&_2^2\&_2^3)^n \quad , \tag{3.12}$$

which we will have occasion to utilize in the sequel, at step 3 of the derivation. □

Step 3. We have so far determined T_3 and T_2, uniquely, and under conditions which do not depend on the probabilistic structure of the problem. In order to complete the derivation of the equilibrium solution, we now consider the minimization problem described by (i) in Def. 2.1, so as to determine T_1.

Towards this end, let us first substitute the unique responses of DM3 and DM2, as given by (3.1) and (3.11), respectively, into J_1, and consider the first Gateaux variation of the resulting quadratic expression around a nominal point $\tilde{\gamma}_1 \in \Gamma_1$:

$$J_1(\gamma_1,\ T_2(\gamma_1),\ T_3[\gamma_1,\ T_2(\gamma_1)]) = E\{\tfrac{1}{2}<\gamma_1,\ \gamma_1>_1 -$$

$$<\gamma_1, (D_{12} + D_{13}D_{32}\&_2^3)(I - K\&_2^2\&_2^3)^{-1}[(D_{21}\&_1^2 + G_1\&_1^2\&_1^3)\gamma_1 + \&_2^2C_2^2z_o$$

$$+ \&_2^2\&_2^3G_2z_o] + D_{13}[D_{31}\&_1^3\gamma_1 + \&_3^3C_3^3z_o] + C_1^1z_o>_1 +$$

$$+ \frac{1}{2}<(I - K\&_2^2\&_2^3)^{-1}[(D_{21}\&_1^2 + G_1\&_1^2\&_1^3)\gamma_1 + \&_2^2C_2^2z_o + \&_2^2\&_2^3G_2z_o],$$

$$(F_{22}^1 - 2F_{23}^1D_{32}\&_2^3)(I - K\&_2^2\&_2^3)^{-1}[(D_{21}\&_1^2 + G_1\&_1^2\&_1^3)\gamma_1 + \&_2^2C_2^2z_o + \&_2^2\&_2^3G_2z_o]$$

$$- 2F_{23}^1[D_{31}\&_1^3\gamma_1 + \&_1^3C_3^3z_o] - 2C_2^1z_o>_2 + \frac{1}{2}<D_{31}\&_1^3\gamma_1 + \&_1^3C_3^3z_o$$

$$+ D_{32}\&_2^3(I - K\&_2^2\&_2^3)^{-1}[(D_{21}\&_1^2 + G_1\&_1^2\&_1^3)\gamma_1 + \&_2^2C_2^2z_o + \&_2^2\&_2^3G_2z_o],$$

$$F_{33}^1[D_{31}\&_1^3\gamma_1 + \&_1^3C_3^3z_o] + F_{33}^1D_{32}\&_2^3(I - K\&_2^2\&_2^3)^{-1}[(D_{21}\&_1^2 + G_1\&_1^2\&_1^3)\gamma_1 + \&_2^2C_2^2z_o$$

$$+ \&_2^2\&_2^3G_2z_o - 2C_3^1z_o>_3 \triangleq L_1(\gamma_1)\ ; \tag{3.13}$$

$$\delta L_1(\tilde{\gamma}_1;h_1) = E\{<h_1(z_1),\ \ell_1(z_1,\ \gamma_1(z_1))>_1\}, \tag{3.14}$$

where

$$\ell_1(z_1,\ \gamma_1(z_1)) = \mathcal{B}\gamma_1(z_1) - (D_{12}\&_2^1 + G_3\&_2^1\&_2^3)k_3(z_2) - k_1(z_1)$$

$$- \&_1^1(\&_1^2D_{21}^* + \&_1^3\&_1^2G_1^*)(I - K\&_2^2\&_2^3)^{-1}\{k_2(z_2) - (F_{22}^1 - G_4\&_2^2\&_2^3)k_3(z_2)\} \tag{3.15}$$

and

$$\mathcal{B} \triangleq I - A_1\&_1^1\&_1^3 - (D_{12}\&_2^1 + G_3\&_2^1\&_2^3)(I - K\&_2^2\&_2^3)^{-1}(D_{21}\&_1^2 + G_1\&_1^2\&_1^3)$$

$$- \&_2^1(D_{21}\&_1^2 + G_1\&_1^2\&_1^3)^*(I - K\&_2^2\&_2^3)^{-1^*}\{(G_4\&_2^2\&_2^3 - F_{22}^1)$$

$$.(I - K\&_2^2\&_2^3)^{-1}(D_{21}\&_1^2 + G_1\&_1^2\&_1^3) + D_{12}^*\&_1^2 + G_3^*\&_1^2\&_1^3\} \tag{3.16a}$$

$$A_1 \triangleq D_{13}D_{31} + D_{31}^*D_{13}^* - D_{31}^*F_{33}^1D_{31} \tag{3.16b}$$

$$G_3 \triangleq D_{13}D_{32} + D_{31}^{*}F_{23}^{1^{*}} - D_{31}^{*}F_{33}^{1}D_{32} \tag{3.16c}$$

$$G_4 \triangleq F_{23}^{1}D_{32} + D_{32}^{*}F_{23}^{1^{*}} - D_{32}^{*}F_{33}^{1}D_{32} \tag{3.16d}$$

$$k_1(z_1) \triangleq C_1^1 E[z_o|z_1] + [D_{13}C_3^3 + D_{31}^{*}C_3^1 - D_{31}^{*}F_{33}^{1}C_3^3]E[E[z_o|z_3]|z_1] \tag{3.17a}$$

$$k_2(z_2) \triangleq C_2^1 E[z_o|z_2] - (D_{32}^{*}C_3^1 + F_{23}^{1}C_3^3 - D_{32}^{*}F_{33}^{1}C_3^3)E[E[z_o|z_3]|z_2] \tag{3.17b}$$

$$k_3(z_2) \triangleq (I - K\mathcal{E}_2^2\mathcal{E}_2^3)^{-1}[\mathcal{E}_2^2 C_2^2 z_o + \mathcal{E}_2^2\mathcal{E}_2^3 G_2 z_o] \quad . \tag{3.17c}$$

This result is obtained through some routine but cumbersome manipulations, and in the course of the derivation we also obtain the second Gateaux variation to be

$$\delta^2 L_1(\tilde{\gamma}_1, h_1) = E\{\langle h_1(z_1), h_1(z_1)\rangle_1 + \langle h_1(z_1), (I - \mathcal{B})h_1(z_1)\rangle_1\} \; . \tag{3.18}$$

Note that both A_1 and G_4 are self-adjoint operators [and so are $A_1\mathcal{E}_1^1\mathcal{E}_1^3$ and $G_4\mathcal{E}_2^2\mathcal{E}_2^3$], and hence the operator $\mathcal{B}$ is self-adjoint, mapping Γ_1 into itself.

The counterpart of inequality (3.3) is also valid here, and by arguing as in step 2, we arrive at the set of necessary and sufficient optimality conditions

$$\delta L_1(\tilde{\gamma}_1; h_1) = 0, \quad \delta^2 L_1(\tilde{\gamma}_1; h_1) \geq 0 \quad \forall\, h_1 \in \Gamma_1, \tag{3.19}$$

which readily leads to the necessary condition [see (3.14)]

$$\ell_1(z_1, \tilde{\gamma}_1(z_1)) = 0 \,, \quad \text{a.e. } \mathcal{P}_{z_1} \,, \tag{3.20}$$

where $\mathcal{P}_{z_1}$ is the probability measure induced under the weak random variable z_1. Let us rewrite (3.20) in a more appealing form, which is the counterpart of (3.6) for DM1:

$$\begin{aligned}\tilde{\gamma}_1(z_1) = {} & (I - \mathcal{B})\tilde{\gamma}_1(z_1) + k_1(z_1) + (D_{12}\mathcal{E}_2^1 + G_3\mathcal{E}_2^1\mathcal{E}_2^3)k_3(z_2) \\ & + \mathcal{E}_1^1(\mathcal{E}_1^2 D_{21}^* + \mathcal{E}_1^3\mathcal{E}_1^2 G_1^*)(I - K\mathcal{E}_2^2\mathcal{E}_2^3)^{-1}\{k_2(z_2) - (F_{22}^1 - G_4\mathcal{E}_2^2\mathcal{E}_2^3)k_3(z_2)\}\end{aligned} \tag{3.21}$$

This is a linear equation in $\tilde{\gamma}_1$, and we now seek conditions, independent of the probabilistic structure of the problem, under which it will admit a unique solution in Γ_1. The general existence condition, which also enables us to compute the solution recursively, as in Proposition 1, is

$$r_1(I - \mathcal{B}) < 1 \tag{3.22}$$

where $r_1(\cdot)$ is defined by (3.9). However, this is dependent on the statistics of the weak random variables involved, and therefore is not precisely the condition that we seek. By appropriate manipulations on (3.22) [details of which are included in the Appendix], we are able to obtain a "deterministic" condition [<u>C2</u>, given below] which insures satisfaction of (3.22). Before presenting this condition, and the general result as Theorem 3.1 below, let us introduce the following expressions:

<u>Preliminary definitions for Condition C2 and Theorem 3.2</u>

$$\begin{aligned}a_1^{(n)} = {} & r_1[D_{12}(K)^n G_1 - D_{21}^* F_{22}^1 (K)^n D_{21} - D_{21}^*(K^*)^n [F_{22}^1 D_{21} - D_{12}^*] \\ & + D_{21}^*(K^*)^{n-1} G_4 G_1 + D_{21}^* G_4 (K_2)^{n-1} D_{21} \\ & - D_{21}^* \sum_{\substack{i=1 \\ i+j=n}}^{n-1} \sum_{j=1}^{n-1} (K^*)^i F_{22}^1 (K)^j D_{21} - D_{21}^* \sum_{\substack{i=1 \\ i+j=n-1}}^{n-1} \sum_{j=1}^{n-1} (K^*)^i G_4 (K)^j D_{21}]\end{aligned} \tag{3.23a}$$

$$a_2^{(n)} = r_1[D_{12}(K)^n G_1 - D_{21}^* F_{22}^1 (K)^n G_1 - D_{21}^* (K^*)^n [F_{22}^1 G_1 - G_3]$$

$$+ D_{21}^* (K^*)^{n-1} G_4 D_{21} + D_{21}^* G_4 (K^*)^{n-1} G_1 \qquad (3.23b)$$

$$- D_{21}^* \sum_{\substack{i=1 \\ i+j=n}}^{n-1} \sum_{j=1}^{n-1} (K^*)^i F_{22}^1 (K)^j G_1 + D_{21}^* \sum_{\substack{i=1 \\ i+j=n-1}}^{n-1} \sum_{j=1}^{n-1} (K^*)^i G_4 (K_2)^j G_1]$$

$$a_3^{(n)} = r_1[G_3(K)^n D_{21} - G_1^* F_{22}^1 (K)^n D_{21} - G_1^* (K^*)^n [F_{22}^1 D_{21} - D_{12}^*]$$

$$+ G_1^* [G_4 (K)^{n-1} D_{21} + (K^*)^{n-1} G_4 G_1] \qquad (3.23c)$$

$$- G_1^* \sum_{\substack{i=1 \\ i+j=n}}^{n-1} \sum_{j=1}^{n-1} (K^*)^i F_{22}^1 (K)^j D_{21} + G_1^* \sum_{\substack{i=1 \\ i+j=n-1}}^{n-1} \sum_{j=1}^{n-1} (K^*)^i G_4 (K)^j D_{21}]$$

$$a_4^{(n)} = r_1[G_3(K)^n G_1 - G_1^* F_{22}^1 (K)^n G_1 - G_1^* (K^*)^n [F_{22}^1 G_1 - G_3] + G_1^* G_4 (K)^{n-1} G_1$$

$$+ G_1^* (K^*)^{n-1} G_4 G_1 - G_1^* \sum_{\substack{i=1 \\ i+j=n}}^{n-1} \sum_{j=1}^{n-1} (K^*)^i F_{22}^1 (K)^j G_1 \qquad (3.23d)$$

$$+ G_1^* \sum_{\substack{i=1 \\ i+j=n-1}}^{n-1} \sum_{j=1}^{n-1} (K^*)^j G_4 (K)^j G_1]$$

$$\bar{r} \triangleq r_1(A_1) + r_1(D_{12} D_{21} + D_{21}^* D_{12}^* - D_{21}^* F_{22}^1 D_{21})$$

$$+ 2r_1(D_{12} G_1 - D_{21}^* [F_{22}^1 G_1 - G_3]) + r_1(G_3 G_1 + G_1^* G_3 - G_1^* F_{22}^1 G_1)$$

$$+ 2r_1(D_{21}^* G_4 G_1) + r_1(D_{21}^* G_4 D_{21}) + r_1(G_1^* G_4 G_1)$$

$$+ \sum_{n=1}^{\infty} [a_1^{(n)} + a_2^{(n)} + a_3^{(n)} + a_4^{(n)}] \qquad (3.24)$$

<u>Condition C2</u> $\quad \bar{r} < 1$

Theorem 3.1 Assume conditions C1 and C2 to hold true.

i) Equation (3.21) admits a unique solution $\gamma_1^o \in \Gamma_1$, which is also the unique solution satisfying (3.19); hence, the decision problem admits a unique hierarchical equilibrium solution, given as $\{\gamma_1^o, \gamma_2^o = T_2(\gamma_1^o), \gamma_3^o = T_3(\gamma_1^o, \gamma_2^o)\}$.

ii) The unique equilibrium strategy γ_1^o of DM1 can be obtained as the convergent limit of an iterative procedure applied to (3.19), which is similar to the one of Proposition 3.1 (ii).

Proof. See the Appendix (Section 6). The iteration is defined explicitly by (6.1). $\sqsupset$

Remark 3.2. The operator $\mathcal{B}$, which plays a crucial role in the solution of (3.21), depends on some inverse operators, specifically on $(I - K\mathcal{E}_2^2\mathcal{E}_2^3)^{-1}$ and its adjoint. However, under C1 each of these operators can be determined as the convergent limit of an infinite series [cf. Remark 3.1, relation (3.12)], and moreover, since $r_2(K\mathcal{E}_2^2\mathcal{E}_2^3) < 1$, a finite truncation of this series will provide a fairly good approximation to $(I - K\mathcal{E}_2^2\mathcal{E}_2^3)^{-1}$. Hence, the linear bounded operator can be written in the form of infinite sums of products of "deterministic" linear bounded operators and some conditional expectations [to see this, simply substitute (3.12) into (3.16a)], and for practical purposes, in the form of finite sums of products of such operators. Therefore, in principle, the iterative procedure of Theorem 3.1 (ii) can be carried out routinely, by performing a series of linear operations. This discussion also readily leads to the following structural result (given in Corollary 3.1 below) in the case of Gaussian distributions. $\sqsupset$

Corollary 3.1. If the weak random variables z_o, z_1, z_2 and z_3 are jointly Gaussian distributed, the unique hierarchical equilibrium solution is affine

in the available information; that is, there exist linear bounded operators $\mathcal{L}_i$; $H_i \to S_i$ $(i = 1, 2, 3)$ and "deterministic" functions $s_i \varepsilon S_i$ $(i = 1, 2, 3)$, such that

$$\gamma_i^o(z_i) = \mathcal{L}_i z_i + s_i \qquad (i = 1, 2, 3) \tag{3.25}$$

Proof. In view of the discussion of Remark 3.2, this result follows from the following two properties of Gaussian weak random variables [cf. Balakrishnan (1976)].

(a) If z_i^1 and z_i^2 are two Gaussian weak random variables which are H_i-valued and σ_i-measurable, and A_1 and A_2 are two "deterministic" linear bounded operators with the same range space, mapping σ_i-measurable H_i-valued variables into σ_i-measurable variables, then the sum $A_1 z_i^1 + A_2 z_i^2$ is also a Gaussian weak random variable.

(b) If z_i and z_j are two Gaussian weak random variables, with the latter being H_j-valued, we have $E[z_i | z_j] = A_{ij} z_j + a_{ij}$, for some linear bounded operator A_{ij} and for some $a_{ij} \varepsilon H_j$.

Repeated utilization of these two properties on the right-hand side of (3.21), for a fixed $\tilde{\gamma}_1(z_1)$ taken as an affine function of z_1, leads to an affine function of z_1 on the left-hand side of (3.21). Since the iteration converges for any starting choice of $\tilde{\gamma}_1$, the statement of the Corollary follows as a special case of Theorem 3.1. □

4. A SPECIAL CASE, AND A NUMERICAL EXAMPLE

Condition C2, which involves some straightforward but rather cumbersome operations in terms of linear operators, simplifies considerably for an important special class of decision problems. Specifically, consider the case when each decision maker interacts (through his cost function) only with the closest decision maker(s) in the hierarchy. In such a case, J_1 will not depend on γ_3, and J_3 will not depend on γ_1, but J_2 will in general depend on both γ_1 and γ_3. Therefore, we now consider the decision problem described by the cost functionals (2.1) - (2.2), but with the terms corresponding to the operators

$$D_{13},\ F^1_{33},\ F^1_{23},\ C^1_3,\ D_{31},\ F^3_{11},\ F^3_{12},\ \text{and}\ C^3_1$$

deleted. Furthermore, we assume that there is no coupling between the decision variables of DM1 and DM3, in the cost function of DM2. Then, the expression corresponding to (3.24) becomes

$$\begin{aligned}\bar{r}' \overset{\Delta}{=}\ & r_1(D_{12}D_{21} + D^*_{21}D^*_{12} - D^*_{21}F^1_{22}D_{21}) + 2r_1(D_{12}G_1 - D^*_{21}F^1_{22}G_1) \\ & + r_1(G^*_1F^1_{22}G_1) + \sum_{n=1}^{\infty} r_1[D^*_{21}\{F^1_{22}(K)^n D_{21} + (K^*)^n(F^1_{22}D_{21} - D^*_{12}) \\ & + \sum_{\substack{i=1 \\ i+j=n}}^{n-1} \sum_{j=1}^{n-1} (K^*)^i F^1_{22}(K)^j D_{21}\}] \ , \qquad (4.1)\end{aligned}$$

which can be obtained by basically following the steps (of the Appendix) that led to (3.24). Hence, what replaces C2 is

C2' $\qquad \bar{r}' < 1$,

under which (and also under C1) the statement of Theorem 3.1 is valid for this special class of decision problems.

As an illustration, consider the case when all spaces are one-dimensional Euclidean, and various parameters assume the values

$$D_{12} = D_{21} = D_{23} = D_{32} = \frac{1}{4}$$

$$F^1_{22} = F^2_{11} = F^2_{33} = F^3_{22} = \frac{1}{8}$$

$$D_{13} = F^1_{33} = F^1_{23} = C^1_3 = D_{31} = F^3_{11} = F^3_{12} = C^3_1 = F^2_{13} = 0 .$$

Condition C1 takes the form

$$r_2(K) = K = 15/128 < 1$$

and therefore it is satisfied. For C2', on the other hand, we have

$$\begin{aligned}\bar{r}' &= \frac{15}{128} + \frac{3}{64}\left(\frac{15}{128}\right) + \frac{1}{128}\sum_{n=2}^{\infty}\left(\frac{15}{128}\right)^n |n-7| \\ &= 0.1171875 + 0.0054931 + 0.0010728 + 0.0001005 + \\ &+ 0.0000088 + 0.0000006 + \ldots . \\ &= 0.1238633 \text{ (to the nearest 7 figures)} \\ &< 1 ,\end{aligned}$$

which indicates that it is also satisfied, and by a comfortably wide margin.

5. EXTENSIONS TO MORE GENERAL TYPES OF DECISION PROBLEMS

The general method of derivation introduced in this paper has been discussed within the context of 3-person stochastic decision problems with linear hierarchy, but it is equally applicable to similarly structured stochastic decision problems with more than 3 decision makers and again with linear hierarchy. A repeated application of the techniques utilized at step 3 in Section 3 (and also in the Appendix), by starting at the bottom of the hierarchy and moving upwards, would lead to a set of conditions (one for each level of hierarchy) which involves multiples of infinite convergent series, under which the hierarchical equilibrium solution would uniquely exist. Furthermore, a natural counterpart of Corollary 3.1 would also hold for this more general problem, in the sense that, under jointly Gaussian statistics, all decision makers' optimal strategies will be affine in their static observations.

Yet another possible extension is to multi-person stochastic decision problems wherein more than one decision maker operates at each level of hierarchy. Then, the single-criterion stochastic optimization adopted in this paper at every level of hierarchy will have to be replaced by multi-criteria stochastic optimization. In particular, if the mode of decision making is noncooperative, and the decision makers (at the same level of hierarchy) adopt the Nash solution concept, a blend of the techniques of this paper and of Başar (1978) can be employed to obtain the hierarchical equilibrium solution and the conditions under which it exists and is unique. If, however, the decision makers at the same level of hierarchy act as members of a team (with a single objective functional), then the theory of Radner (1962) will have to be used as a supplementary technique

in the derivation. In either case, when the underlying statistics are Gaussian, the optimal policy for each decision maker will be in the form (3.25).

6. APPENDIX

In this appendix, we provide a proof for Theorem 3.1.

We have already shown in the discussion that precedes the statement of the theorem, that the first condition of (3.19) is satisfied if and only if (3.21) admits a solution, and that this solution exists and is unique if (3.22) is satisfied.

Taking this as our starting point, we first substitute (3.12) in (3.16a) to obtain

$$
\begin{aligned}
I - \mathcal{B} = {} & A_1 \mathcal{E}_1^1 \mathcal{E}_1^3 + (D_{12}\mathcal{E}_2^1 + G_3 \mathcal{E}_2^1 \mathcal{E}_2^3)][I + \sum_{n=1}^{\infty} (K\mathcal{E}_2^2\mathcal{E}_2^3)^n](D_{21}\mathcal{E}_1^2 + G_1 \mathcal{E}_1^2 \mathcal{E}_1^3) \\
& + \mathcal{E}_2^1(\mathcal{E}_1^2 D_{21}^* + \mathcal{E}_1^3 \mathcal{E}_1^2 G_1^*) \; [I + \sum_{n=1}^{\infty} (\mathcal{E}_2^3 \mathcal{E}_2^2 K^*)^n]\{(G_4 \mathcal{E}_2^2 \mathcal{E}_2^3 - F_{22}^1) \\
& [I + \sum_{n=1}^{\infty} (K\mathcal{E}_2^2\mathcal{E}_2^3)^n](D_{21}\mathcal{E}_1^2 + G_1 \mathcal{E}_1^2 \mathcal{E}_1^3) + D_{12}^* \mathcal{E}_1^2 + G_3^* \mathcal{E}_1^2 \mathcal{E}_1^3\} ,
\end{aligned}
$$

where we have utilized the following property of adjoints of conditional expectations

$$
(\mathcal{E}_2^2 \mathcal{E}_2^3)^* = \mathcal{E}_2^{3^*} \mathcal{E}_2^{2^*} = \mathcal{E}_2^3 \mathcal{E}_2^2 .
$$

Now, multiplying out several terms, and collecting some common terms together, we arrive at

$$
\begin{aligned}
I - \mathcal{B} = {} & A_1 \mathcal{E}_1^1 \mathcal{E}_1^3 + [D_{12}D_{21} + D_{21}^* D_{12}^* - D_{21}^* F_{22}^1 D_{21}] \mathcal{E}_1^1 \mathcal{E}_1^2 \\
& + [D_{12}G_1 - D_{21}^*(F_{22}^1 G_1 - G_3)] \mathcal{E}_1^1 \mathcal{E}_1^2 \mathcal{E}_1^3 + [G_3 D_{21} + G_1^* D_{12}^* - G_1^* F_{22}^1 D_{21}] \mathcal{E}_1^1 \mathcal{E}_1^3 \mathcal{E}_1^2 \\
& + [G_3 G_1 + G_1^* G_3^* - G_1^* F_{22}^1 G_1] \mathcal{E}_1^1 \mathcal{E}_1^3 \mathcal{E}_1^2 \mathcal{E}_1^3 + D_{21}^* G_4 D_{21} \mathcal{E}_1^1 \mathcal{E}_1^2 \mathcal{E}_1^3 \mathcal{E}_1^2 \\
& + D_{21}^* G_4 G_1 \mathcal{E}_1^1 (\mathcal{E}_1^2 \mathcal{E}_1^3)^2 + G_1^* G_4 D_{21} \mathcal{E}_1^* (\mathcal{E}_1^3 \mathcal{E}_1^2)^2 + G_1^* G_4 G_1 \mathcal{E}_1^1 \mathcal{E}_1^3 (\mathcal{E}_1^2 \mathcal{E}_1^3)^2 \\
& + \sum_{n=1}^{\infty} [\Lambda_1^{(n)} \mathcal{E}_1^1 (\mathcal{E}_1^2 \mathcal{E}_1^3)^n \mathcal{E}_1^2 + \Lambda_2^{(n)} \mathcal{E}_1^1 (\mathcal{E}_1^2 \mathcal{E}_1^3)^{n+1} + \Lambda_3^{(n)} \mathcal{E}_1^1 (\mathcal{E}_1^3 \mathcal{E}_1^2)^{n+1}
\end{aligned}
$$

$$+ \Lambda_4^{(n)}\mathcal{E}_1^1\mathcal{E}_1^3(\mathcal{E}_1^2\mathcal{E}_1^3)^{n+1}]$$

where Λ_1 denotes the linear bounded operator inside the brackets on the right-hand side of (3.23a), and Λ_2, Λ_3, Λ_4 are defined likewise through (3.23b) - (3.23d).

For two linear bounded operators $\mathcal{A}_1$ and $\mathcal{A}_2$ mapping Γ_1 into Γ_1, we have [cf. Kantorovich and Akhilov (1977)]

$$r_1(\mathcal{A}_1 + \mathcal{A}_2) \leq r_1(\mathcal{A}_1) + r_2(\mathcal{A}_2) ,$$

and therefore the spectral radius of $I - \mathcal{B}$ can be bounded from above by

$$\begin{aligned} r_1(I - \mathcal{B}) \leq\ & r_1(A_1\mathcal{E}_1^1\mathcal{E}_1^3) + r_1[(D_{12}D_{21} + D_{21}^*D_{12}^* - D_{21}^*F_{22}^1D_{21})\mathcal{E}_1^1\mathcal{E}_1^2] \\ & + r_1[(D_{12}G_1 - D_{21}^*(F_{22}^1G_1 - G_3))\mathcal{E}_1^1\mathcal{E}_1^2\mathcal{E}_1^3] \\ & + r_1[(G_1^*D_{12}^* - (G_1^*F_{22}^1 - G_3^*)D_{21})\mathcal{E}_1^1\mathcal{E}_1^3\mathcal{E}_1^2] \\ & + r_1[(G_3G_1 + G_1^*G_3^* - G_1^*F_{22}^1G_1)\mathcal{E}_1^1\mathcal{E}_1^3\mathcal{E}_1^2\mathcal{E}_1^3] \\ & + r_1[D_{21}^*G_4D_{21}\mathcal{E}_1^1\mathcal{E}_1^2\mathcal{E}_1^3\mathcal{E}_1^2] + r_1[D_{21}^*G_4G_1\mathcal{E}_1^1(\mathcal{E}_1^2\mathcal{E}_1^3)^2] \\ & + r_1[G_1^*G_4D_{21}\mathcal{E}_1^*(\mathcal{E}_1^3\mathcal{E}_1^2)^2] + r_1[G_1^*G_4G_1\mathcal{E}_1^1\mathcal{E}_1^3(\mathcal{E}_1^2\mathcal{E}_1^3)^2] \\ & + \sum_{n=1}^{\infty}\{r_1[\Lambda_1^{(n)}\mathcal{E}_1^1(\mathcal{E}_1^2\mathcal{E}_1^3)^n\mathcal{E}_1^2] + r_1[\Lambda_2^{(n)}\mathcal{E}_1^1(\mathcal{E}_1^2\mathcal{E}_1^3)^{n+1}] \\ & + r_1[\Lambda_3^{(n)}\mathcal{E}_1^1(\mathcal{E}_1^3\mathcal{E}_1^2)^{n+1}] + r_1[\Lambda_4^{(n)}\mathcal{E}_1^1\mathcal{E}_1^3(\mathcal{E}_1^2\mathcal{E}_1^3)^{n+1}]\} . \end{aligned}$$

Now, utilizing the line of argument that led to inequality (i) in the proof of Proposition 3.1, we may readily factor out the projection operators corresponding to conditional expectations, and thus obtain the final bound

$$r_1(I - \mathcal{B}) \leq \bar{r}$$

where $\bar{r}$ is given by (3.24), and we have used the property that $r_1(\mathcal{A}) = r_1(\mathcal{A}^*)$ for any linear bounded operator $\mathcal{A}: \Gamma_1 \to \Gamma_1$. Then, condition <u>C2</u> guarantees, in view of our discussion in the proof of Proposition 3.1, existence of a unique solution to (3.21), which can be obtained as the convergent limit of the series generated by the iterative procedure

$$\begin{aligned}\gamma_1^{(n+1)} = (I - \mathcal{B})\gamma_1^{(n)} + k_1 + (D_{12}\mathcal{E}_2^1 + G_3\mathcal{E}_2^1\mathcal{E}_2^3)k_3 \\ + \mathcal{E}_1^1(\mathcal{E}_1^2D_{21}^* + \mathcal{E}_1^3\mathcal{E}_1^2G_1^*)(I - K\mathcal{E}_2^2\mathcal{E}_2^3)^{-1}\{k_2 - (F_{22}^1 - G_4\mathcal{E}_2^2\mathcal{E}_2^3)k_3\}\end{aligned} \tag{6.1}$$

starting at any initial choice $\gamma_1^{(0)} \in \Gamma_1$.

To complete the proof of the Theorem, we now show that the second Gateaux variation (3.18) is positive definite under <u>C2</u>.

Firstly, since $\bar{r} < 1$, and $\mathcal{B}$ is self-adjoint,

$$|E\{\langle h_1(z_1), (I - \mathcal{B})h_1(z_1)\rangle_1\}| < E\{\langle h_1(z_1), h_1(z_1)\rangle_1\} ,$$

unless h_1 is the zero element in Γ_1. Using this in (3.18) we have

$$\delta^2 L_1 > E\{\langle h_1(z_1), h_1(z_1)\rangle_1 - \langle h_1(z_1), h_1(z_1)\rangle_1\} = 0 \quad ,$$

provided that h_1 is not the zero element. Hence, $\delta^2 L_1$ is <u>positive definite</u>, which completes the proof of the theorem.

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THE GAUSSIAN TEST CHANNEL WITH AN INTELLIGENT JAMMER*

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ABSTRACT

Consider the problem of transmitting a sequence of identically distributed independent Gaussian random variables through a Gaussian memoriless channel with a given input power constraint, in the presence of an intelligent jammer. The jammer taps the channel and feeds back a signal, at a given energy level, for the purpose of jamming the transmitting sequence. Under a square-difference distortion measure which is sought to be maximized by the jammer and minimized by the transmitter and the receiver, this paper obtains the complete set of optimal (saddle-point) policies. The solution is essentially unique, and it is structurally different in three different regions in the parameter space, which are determined by the signal-to-noise ratios and relative magnitudes of the noise variances. The best (maximin) policy of the jammer is either to choose a linear function of the measurement he receives through channel-tapping, or to choose, in addition (and additively), an independent Gaussian noise sequence, depending on the region where the parameters lie. The optimal (minimax) policy of the transmitter is to amplify the input sequence to the given power level by a linear transformation, and that of the receiver is to use a Bayes estimator.

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I. INTRODUCTION AND PROBLEM DESCRIPTION

The communication system depicted in Fig. 1 represents an extended version of the so-called Gaussian test channel (cf. [1]), which also includes an intelligent jammer who has access to a (possibly) noise-corrupted version of the signal to be transmitted through a Gaussian channel. More specifically, a Gaussian random variable† of zero mean and unit variance [denoted $u \sim N(0,1)$] is to be transmitted through a Gaussian channel with input energy constraint c^2, and additive noise ($w = w_1 + w_2$) with total noise variance $\xi = \xi_1 + \xi_2$. Let the transmitter strategy be denoted $\gamma(\cdot)$, which is an element of the space Γ_t of real-valued Borel measurable functions satisfying the power constraint $E\{[\gamma(u)]^2\} \le c^2$. The jammer has access to a noise-corrupted version of

$$x \triangleq \gamma(u) + w_1 \quad , \tag{1}$$

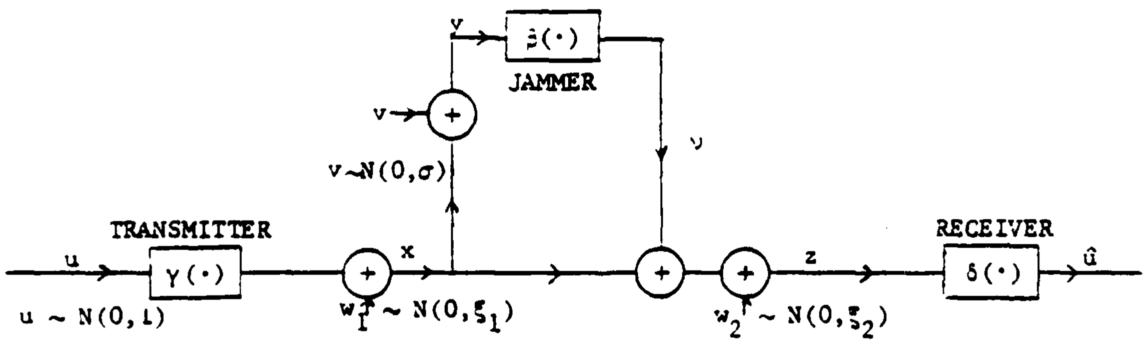


Fig. 1. The Gaussian test channel with an intelligent jammer

†This single variable can be replaced with a sequence of independent identically distributed Gaussian random variables, without altering the results of this paper.

denoted

$$y = x + v \quad , \tag{2}$$

where $v \sim N(0,\sigma)$, all random variables (u, w_1, w_2 and v) are statistically independent, and $\xi_1 \geq 0$, $\xi_2 \geq 0$ and $\sigma \geq 0$. Based on the observed value of y, the jammer feeds back a second-order random variable $\nu = \beta(y)$ to the channel, so that the input to the receiver is now

$$z = x + \nu + w_2 \quad . \tag{3}$$

The random variable ν is correlated with y, but it is not necessarily determined through a deterministic transformation on y [i.e. $\beta(\cdot)$ is in general a random mapping]; furthermore it satisfies the energy constraint $E[\nu^2] \leq k^2$. Let us denote the class of all associated probability measures μ for the jammer by M_j. Finally, the receiver applies a Borel-measurable transformation $\delta(\cdot)$ on its input z, so as to produce an estimate $\hat{u}$ of u, by minimizing the square-difference distortion measure

$$R(\gamma, \delta, \mu) = \int_{-\infty}^{\infty} E\{[\delta(z) - u]^2 \mid \nu\} \, d\mu(\nu) \quad . \tag{4}$$

Denote the class of all Borel-measurable mappings $\delta(\cdot)$, to be used as an estimator for u, by Γ_r. Then, the transmitter and the receiver seek to minimize R by a proper choice of $\gamma \in \Gamma_t$ and $\delta \in \Gamma_r$, respectively, and the jammer seeks to maximize the same quantity by his choice of $\mu \in M_j$. Since there is a complete conflict of interests in this communication problem, an "optimal" transmitter-receiver-jammer policy would be the saddle-point solution ($\gamma^* \in \Gamma_t$, $\delta^* \in \Gamma_r$, $\mu^* \in M_j$) satisfying the set of inequalities

$$R(\gamma^*, \delta^*, \mu) \leq R(\gamma^*, \delta^*, \mu^*) \leq R(\gamma, \delta, \mu^*) \quad , \quad \forall \gamma \in \Gamma_t, \delta \in \Gamma_r, \mu \in M_j. \tag{5}$$

The maximin policy $\_^*$ is also known as a least-favorable probability measure for the jammer [3].

In this paper, we verify existence and "essential" uniqueness[+] of the saddle-point solution, and determine the corresponding policies explicitly and in analytic form. The main result is presented in the next section and, in particular, in Theorem 1. The structure of the solution is different in three different regions of the parameter space; in one of these regions the solution is trivial, and in the other two regions (which are covered by Theorem 1) the saddle-point policy for the transmitter is to amplify the input signal to the maximum power level through a linear transformation. The saddle-point policy for the jammer is to choose a Gaussian random variable (or a sequence of independent identically distributed Gaussian random variables, if the input is also a sequence) which is correlated with the input signal; the nature of this correlation turns out to be different in the two regions of interest. For the receiver, the optimal policy is to use a Bayes estimator. The proof of this result, which is given in section II, is rather involved, and at places it requires some rather intricate arguments, but it is essentially a proof of the "verification" type.

Section III of the paper includes some discussion on special cases and on some related results in the literature. The Appendix provides proofs of two Lemmas which are utilized in the derivation in section II.

[+] This term will be made clear in the next section.

II. DERIVATION OF THE SADDLE-POINT SOLUTION

In this section, we obtain the saddle-point solution of the problem formulated in Section I, for all values of the parameters, $c \geq 0$, $k \geq 0$, $\xi_1 \geq 0$, $\xi_2 \geq 0$, $\sigma \geq 0$. There exists, however, a region in this parameter space, in which the problem is trivial, in the sense that the jammer has the power to do the best he can possibly do, by cancelling out the signal component $\gamma(x)$ in the received signal z. Specifically, consider the region

<u>R1</u>.
$$k^2 \geq c^2 + \xi_1 + \sigma \quad ,$$

where the deterministic feedback policy

$$\beta^*(y) = -y \tag{6}$$

is feasible for the jammer, which leads to

$$z = w_2 - v$$

and thereby to

$$\delta^*(z) = 0 \quad , \tag{7}$$

resulting in a maximum distortion level of

$$R(\gamma, \delta^*, \beta^*) = 1 \quad .$$

Note that the choice of any specific coding strategy is irrelevant here, since they all lead to the same maximum distortion level, under (6) and (7). Hence, for this special case, the pair (δ^*, β^*) as given by (6) - (7) constitutes a (trivial) saddle-point solution (and the only one) for any choice of $\gamma \in \Gamma_t$.

Leaving this "uninteresting" case aside, we henceforth restrict our analysis to the parameter region

R2. $$k^2 < c^2 + \xi_1 + \sigma \quad ,$$

which we further decompose into two subregions characterized by the additional constraints

R3. $$k^2 - \frac{(c^2 + \xi_1)\,k}{(c^2 + \xi_1 + \sigma)^{\frac{1}{2}}} + \xi_2 > 0$$

R4. $$k^2 - \frac{(c^2 + \xi_1)\,k}{(c^2 + \xi_1 + \sigma)^{\frac{1}{2}}} + \xi_2 \leq 0 \quad .$$

The complete solution to the problem is now provided in Theorem 1 below, after introducing some notation and terminology.

Preliminary notation for Theorem 1

Introduce the scalar parameters λ and t by

$$\lambda = -\,k/(c^2 + \xi_1 + \sigma)^{\frac{1}{2}} \tag{8a}$$

$$t = 1 - \{(k^2 + \xi_2)^2 (c^2 + \xi_1 + \sigma)/[k^2 (c^2 + \xi_1)^2]\} \quad , \tag{8b}$$

and let η denote a Gaussian random variable with mean zero and variance tk^2, i.e.,

$$\eta \sim N(0,\ tk^2) \tag{9}$$

whenever $t \geq 0$.

Theorem 1. In region R2, the communication problem admits two saddle-point solutions $(\gamma^*, \delta^*, \mu^*)$ and $(-\gamma^*, -\delta^*, \mu^*)$, where

(i) $\gamma^*(u) = cu,$ (10)

(ii) μ^* is the Gaussian probability measure associated with the random variable

$$\nu = \beta^*(y) = \begin{cases} \lambda y, & \text{in } \underline{R3} \\ \lambda(1-t)^{\frac{1}{2}} y + \eta, & \text{in } \underline{R4} \end{cases} \tag{11}$$

where $t \in [0,1]$ in R4, and $\eta \sim N(0, tk^2)$.

(iii) δ^* is the Bayes estimator for u under the least favorable distribution μ^*, computed as

$$\delta^*(z) = \begin{cases} \{c(1+\lambda)/[(1+\lambda)^2(c^2+\xi_1) + \lambda^2\sigma + \xi_2]\}z, & \text{in } \underline{R3} \\ [c/(c^2 + \xi_1)]z, & \text{in } \underline{R4}. \end{cases} \tag{12}$$

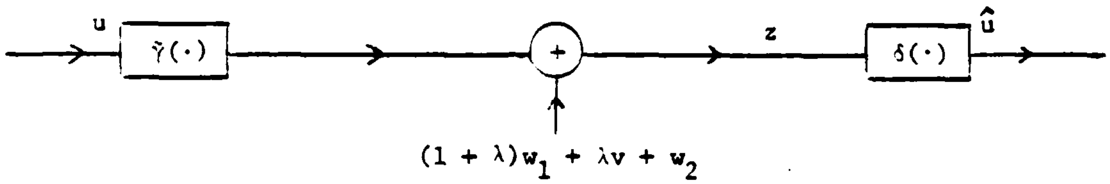
Proof. The proof proceeds in two steps. We first establish validity of the right-hand side (RHS) inequality of (5) when μ^* is determined by (11), and then prove the left-hand side (LHS) inequality of (5) when γ^* and δ^* are given by (10) and (12), respectively. Finally we discuss the "essential uniqueness" property of the saddle-point solution.

(a) THE RHS INEQUALITY

Region R2 $\cap$ R3: Suppose that μ^* is determined by (11) and the parameter values lie in region R2 $\cap$ R3. Then, the RHS inequality of (5) dictates a combined coding-decoding problem, with the channel output [equivalently, receiver input] being [from (3)]

$$z = (1 + \lambda)\gamma(u) + (1 + \lambda)w_1 + \lambda v + w_2,$$

where $0 < (1+\lambda) < 1$ from (8a), since we are in region R2. Let $\tilde{\gamma}(u) \triangleq (1+\lambda)\gamma(u)$. Then, the problem we face is the Gaussian test channel



with square-difference distortion, Gaussian channel noise [with mean zero and variance $(1+\lambda)^2\xi_1 + \lambda^2\sigma + \xi_2$], and channel-input energy constraint

$$E\{[\tilde{\gamma}(u)]^2\} \le (1+\lambda)^2 c^2 \quad .$$

It is well known that this problem admits a linear solution [cf. [1]], which is that the best coding scheme is to amplify the input u to the maximum available power level, i.e.

$$\tilde{\gamma}^*(u) = c(1+\lambda)u \quad , \quad c > 0 \quad ,$$

and to choose the quadratic distortion minimizer δ as the Bayes estimator

$$\delta^*(z) = E[u|z] = \{(1+\lambda)c/[(1+\lambda)^2(c^2+\xi_1) + \lambda^2\sigma + \xi_2]\}z$$

which is precisely (12) in the region R2 $\cap$ R3. Moreover, since $1+\lambda > 0$,

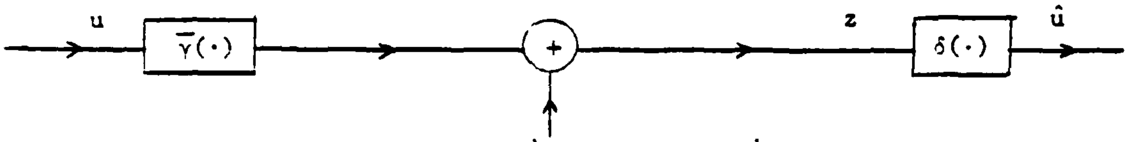
$$\gamma^*(u) = \frac{1}{1+\lambda}\tilde{\gamma}^*(u) = cu \quad ,$$

which is the same as (10). Hence, we have established the validity of the RHS of (5), for the solution presented in Theorem 1, in the region R2 $\cap$ R3. Note that, yet another possible coding policy for the transmitter would be

$$\gamma^*(u) = -cu \quad , \quad c > 0 \quad ,$$

i.e. amplification by a negative factor, but since this leads to the same minimum distortion level, we adopt the convention of choosing only the positive amplification factor and call such a solution "essentially unique".

Region $R2 \cap R4$: Now suppose that the parameter values lie in the region $R2 \cap R4$. A similar reasoning as above again leads to a Gaussian test channel



$$w = [1 + \lambda(1-t)^{\frac{1}{2}}]w_1 + \lambda(1-t)^{\frac{1}{2}}v + \eta + w_2$$

with the channel-input energy constraint being

$$E\{[\bar{\gamma}(u)]^2\} \leq c^2[1 + \lambda(1-t)^{\frac{1}{2}}]^2 \quad ,$$

where $\bar{\gamma}(u) \triangleq [1 + \lambda(1-t)^{\frac{1}{2}}]\gamma(u)$. The Gaussian channel noise has its mean zero and variance

$$\operatorname{var}(w) = [1 + \lambda(1-t)^{\frac{1}{2}}]^2\xi_1 + \lambda^2(1-t)\sigma + tk^2 + \xi_2 \quad .$$

Substituting for λ and t from (8a) and (8b), respectively, we can evaluate the latter expression to be

$$\operatorname{var}(w) = \xi_1 - \xi_2 - k^2 - \frac{(k^2+\xi_2)^2c^2}{(c^2+\xi_1)^2} + \frac{2(k^2+\xi_2)c^2}{(c^2+\xi_1)} \quad .$$

Likewise, the input power constraint can be written as

$$\operatorname{var}(\bar{\gamma}(u)) \leq c^2[c^2 + \xi_1 - \xi_2 - k^2]^2/(c^2+\xi_1)^2 \triangleq m$$

and furthermore

$$m + \mathrm{var}\ (w) = c^2 + \xi_1 + \xi_2 - k^2 \ .$$

Again using the well-known result for Gaussian test channels, we obtain the essentially unique solution to be

$$\bar{\gamma}^*(u) = c[1 + \lambda(1 - t)^{1/2}]u$$

$$\delta^*(z) = E[u|z] = \frac{E[uz]}{\mathrm{var}\ z} z = \frac{m^{1/2}}{m + \mathrm{var}(w)} z = \frac{c}{c^2 + \xi_1} z \ ,$$

with the latter expression verifying (12) in region R4. Furthermore, since $1 + \lambda(1 - t)^{1/2}$ is nonsingular (in fact, it lies in (0,1] under the assertion that $t \in [0,1]$),

$$\gamma^*(u) = \bar{\gamma}^*(u)/[1 + \lambda(1 - t)^{1/2}]$$
$$= cu$$

which verifies (10), also in the region R4. This then completes the proof of parts (i) and (iii) of the Theorem, under the assertions that a least favorable distribution μ^* for ν exists, as given by (11), and $t \in [0,1]$ in R4. The former assertion is verified next, in the sequel, and the latter one is verified in Lemma 2 in the Appendix.

(b) THE LHS INEQUALITY

Region R2 $\cap$ R3: Suppose now that γ^* and δ^* are given by (10) and (12), respectively, and the parameter values lie in R2 $\cap$ R3. Then, the LHS inequality of (5) dictates the following optimization problem for the jammer:

$$\max_{\mu \in M_j} \int_{-\infty}^{\infty} E\{[\alpha z - u]^2 | \nu\}\, d\mu(\nu) \ , \tag{13}$$

where

$$z = cu + w_1 + \nu + w_2 \quad ,$$

$$\alpha = c(1+\lambda)/[(1+\lambda)^2(c^2 + \xi_1) + \lambda^2\sigma + \xi_2] \quad . \tag{14}$$

Note that (13) can also be written as

$$\max_{\mu \in M_j} \int_{-\infty}^{\infty} E_y\{E\{[\alpha z - u]^2 |\nu, y\}\} \, d\mu(\nu) = \max_{\mu \in M_j} E_y\{\int_{-\infty}^{\infty} E\{\alpha^2 z^2 - 2\alpha z u |\nu, y\} d\mu(\nu|y)\} + 1 \; .$$

where

$$y = cu + w_1 + \nu.$$

Furthermore, since w_2 is independent of u, w_1 and ν, u is independent of w_1, and they both have zero mean, the latter expression can be simplified to

$$\max_{\mu \in M_j} E_y\{ \int_{-\infty}^{\infty} E\{\alpha^2\nu^2 + \alpha^2 2\nu(cu + w_1) + \alpha^2(c^2u^2 + w_1^2 + w_2^2) - 2\alpha cu^2$$

$$- 2\alpha\nu u|\nu, y\} \, d\mu(\nu|y)\} + 1$$

$$= \max_{\mu \in M_j} E_y\{ \int_{-\infty}^{\infty} [\alpha^2\nu^2 - 2\nu\pi(y)] \, d\mu(\nu|y)\} + (\alpha c - 1)^2 + \alpha^2(\xi_1 + \xi_2)$$

where

$$\pi(y) = \alpha E[u|y] - \alpha^2 c E[u|y] - \alpha^2 E[w_1|y]$$

$$= \{[\alpha(1 - \alpha c)c/(c^2 + \xi_1 + \sigma)] - [\alpha^2 \xi_1/(c^2 + \xi_1 + \sigma)]\}y$$

$$= [-\alpha^2 + \alpha(c + \alpha\sigma)/(c^2 + \xi_1 + \sigma)]y \triangleq py \quad . \tag{15}$$

Hence, we may confine attention to the maximization problem

$$J = \max_{\mu \in M_j} E_y\{ \int_{-\infty}^{\infty} [\alpha^2\nu^2 - 2\nu\pi(y)] \, d\mu(\nu|y)\} \quad , \tag{16a}$$

which is in fact invariant under the transformation $c \to -c$, and is therefore also the maximization problem (for the jammer) corresponding to the pair $(-\gamma^*, -\delta^*)$. Now, by utilizing the Cauchy-Schwartz inequality [cf. [2]], (16a) can be bounded from above by

$$J \leq \max_{\mu \in M_j} \{ \int_{-\infty}^{\infty} \alpha^2 \nu^2 \, d\mu(\nu) + 2 \left[\int_{-\infty}^{\infty} \nu^2 d\mu(\nu) \right]^{1/2} |E_y\{|\pi(y)|^2\}|^{1/2} \}$$

and since $\mu \in M_j$, this can further be bounded from above by

$$J \leq \alpha^2 k^2 + 2k|E_y\{|\pi(y)|^2\}|^{1/2} \quad . \tag{16b}$$

But, provided that

$$E_y\{[|\pi(y)|^2]\} \neq 0 \quad ,$$

this upper bound is attained <u>uniquely</u> if we choose, in (16a), $\mu(\nu|y)$ to be the one-point conditional probability measure corresponding to the strategy

$$\nu = \beta^*(y) = -[k/[E_y\{|\pi(y)|^2\}]^{1/2}] \; \pi(y) \quad , \tag{17}$$

which may be verified by direct substitution of (17) into (16a) and by comparing the resulting expression with the upper bound (16b). Now, what remains to be shown is that (17) is equivalent to (11) in region $\underline{R2} \cap \underline{R3}$, and that $E_y\{|\pi(y)|^2\} > 0$. Lemma 1 in the Appendix proves that the coefficient p of **y** in (15) is in fact positive in the region $\underline{R2} \cap \underline{R3}$, and hence the latter requirement is readily fulfilled. Furthermore, since

$$\frac{-k}{[E_y\{|\pi(y)|^2\}]^{1/2}} \pi(y) = \frac{-kp}{|p| \, |\mathrm{var}\ (y)|^{1/2}} \, y = \lambda y \quad ,$$

from (8a) and the property that $p > 0$, the former requirement is also satisfied. This then completes the verification of the LHS inequality of

(5), and thereby verification of the theorem, for the region R2 ^ R3.

Region R2 ^ R4: We now finally verify the LHS inequality of (5) when the parameters belong to the region R2 ^ R4. What replaces the maximization problem (13) in this case is

$$\max_{\mu \in M_j} \int_{-\infty}^{\infty} E\{[\frac{c}{c^2 + \xi_1} z - u]^2 | \nu\} \, d\mu(\nu) \quad ,$$

which can be rewritten as (through some straightforward manipulations)

$$\frac{c^2}{(c^2 + \xi_1)^2} \left\{ \max_{\mu \in M_j} E_y[\int_{-\infty}^{\infty} \{\nu^2 - 2\nu E[\frac{\xi_1}{c} u - w_1 | y]\} \, d\mu(\nu|y)] + \xi_1 + \xi_2 + \frac{\xi_1^2}{c^2} \right\}, \tag{18}$$

which is an expression that is invariant under the transformation $c \to -c$. Now, note that

$$E[\frac{\xi_1}{c} u - w_1 | y] = \frac{\xi_1}{c} \cdot \frac{c}{c^2 + \xi_1 + \sigma} y - \frac{\xi_1}{c^2 + \xi_1 + \sigma} y = 0 \ ,$$

and therefore the maximizing solution is any probability measure μ, with the property

$$\int_{-\infty}^{\infty} \nu^2 \, d\mu(\nu) = k^2 \quad .$$

Let us now investigate whether μ^*, determined by (11), is one such measure in region R2 ^ R4. Towards this end, it suffices to show that

$$\mathrm{var}[\lambda(1 - t)^{1/2} y + \eta] = k^2$$

and

$$t \in [0,1] \quad .$$

The latter is shown in Lemma 2, in the Appendix. For the former, simply note that, because $\eta \sim N(0, tk^2)$ and independent,

$$\begin{aligned} \text{var } [\lambda(1-t)^{1/2}y + \eta] &= \lambda^2(1-t) \text{ var } (y) + \text{var } (\eta) \\ &= \lambda^2(1-t)(c^2 + \xi_1 + \sigma) + tk^2 \\ &= k^2(1-t) + tk^2 = k^2 \quad , \end{aligned}$$

thus establishing the desired result. As a parenthetical remark, we should mention that the estimator (12) in region R2 $\cap$ R4 may also be viewed as an equalizer decision rule [see [3]] since the conditional (on ω) risk function corresponding to it is a constant on ∂M_j, the boundary of M_j. [Note that in this interpretation, elements of ∂M_j are the decision variables of the jammer, and we *have to introduce probability measures* on ∂M_j.] Hence, the minimax (saddle-point) property of δ^* in R2 $\cap$ R4 can also be verified [with γ^* fixed, as given] by resorting to a well-known property of equalizer decision rules when they are also Bayes with respect to a least favorable probability measure [which in this case is the one-point distribution on ∂M_j, which selects the Gaussian random variable $\lambda(1-t)^{1/2}y + \eta$]; see, [4], [5]. But, the proof given here seems to be more suited to the problem under consideration since (i) it does not require additional probability measures to be defined on M_j, and (ii) it also establishes the optimality of γ^*.

To recapitulate, we have verified existence of a saddle-point solution (10) - (12) for the communication problem under consideration, in the parameter region R2. The analysis also readily leads to the conclusion that in addition to (10) - (12), the triple $(-\gamma^*, -\delta^*, \omega^*)$ also provides a saddle-point solution, naturally leading to the same saddle-point value

for R. The question now arises as to whether other saddle-point equilibria exist. In region R2 $\cap$ R3, there is clearly no other saddle point, since the maximization problem (16a) [which corresponds to both (γ^*, δ^*) and $(-\gamma^*, -\delta^*)$] admits a unique solution, thereby eliminating the possibility of multiple saddle-point policies for the jammer. [Otherwise, interchangeability property of saddle points (cf. [6]) would lead to a contradiction]. In the remaining part of R2, i.e. R2 $\cap$ R4, however, the issue is more subtle. Since the maximization problem (18) is invariant under different choices of probability measures from ∂M_j, the LHS inequality of (5) clearly does not admit a unique solution - in fact, all second-order probability measures with first moment zero and second moment equal to k^2 constitute a solution. But, for any one of these to constitute a saddle-point policy for the jammer, it has to be in equilibrium with (γ^*, δ^*), because of the interchangeability property of saddle-point equilibria. This further implies that, with γ^* fixed as given, δ^* has to be Bayes with respect to that least-favorable distribution. Since δ^* is a linear estimator and all random variables are Gaussian, this requires the chosen element of ∂M_j to be a Gaussian probability measure, and some further analysis reveals that (11) is in fact the only such element. $\square$

Some of the expressions derived in the proof of Theorem 1 now lead to the following Corollary which gives the saddle-point values in different regions.

Corollary 1. The saddle-point value (R^*) of $R(\gamma, \delta, \mu)$ in different regions is given as follows:

R1: $\quad R^* = 1$

<u>R2</u> $\cap$ <u>R3</u>: $$R^* = (\alpha c - 1)^2 + \alpha^2(\xi_1 + \xi_2 + k^2) + 2kp^2(c^2 + \xi_1 + \sigma)$$

<u>R2</u> $\cap$ <u>R4</u>: $$R^* = \frac{c^2}{(c^2 + \xi_1)^2}\left(k^2 + \xi_1 + \xi_2 + \frac{\xi_1^2}{c^2}\right)$$

where α and p are defined by (14) and (15), respectively. $\square$

III. DISCUSSION OF SOME SPECIAL CASES, AND CONCLUDING REMARKS

The general solution to the communication problem of Fig. 1 has the property that it is structurally different in the two regions of interest, with the dividing "line" between these two regions being a hyperplane determined by the allowable power levels for the transmitter and jammer, and the noise intensities in the main channel and the jammer's wiretap link. In particular, if the transmitter's allowable power level (c^2) is larger than that of the jammer (k^2), we stay in region R2, and if this difference is sufficiently large the jammer's maximin policy is an additive mixture of a linear transformation on his measurement and an independent Gaussian random variable, whereas if the difference is small it is more likely (depending on the values of other parameters) that his maximin policy will be only a linear transformation on his measurement.

If the wiretapping channel noise variance (σ) is sufficiently large, the parameter region is R2 ∪ R3, and hence the optimum strategy for the jammer is a linear policy - which may seem, at first sight, to be somewhat counter-intuitive, since the information contained in y (concerning u) is quite unreliable. However, some scrutiny reveals that the jammer, in fact, uses this noisy measurement as a source of noise in order to jam the transmission channel. This makes particular sense in the limiting case $\sigma \to \infty$, when the optimal jamming policy is to choose μ^* as a Gaussian distribution with mean zero and variance k^2, which should be independent of the transmitter output. This conclusion for the limiting case corroborates a result obtained in [7] in a somewhat different context. More specifically, this recent reference addresses the problem of obtaining optimal policies in the presence of jamming, when jammer's policies (considered as random variables) are forced

to be independent of the transmitter outputs, and the loss function (to be minimaximized) is taken as the mutual information between the transmitter output and the receiver input. In this framework, McEliece and Stark solve in [7], as an application of their general approach, the communication problem depicted in Fig. 1, but without the tapping channel, and arrive at the conclusion that the least-favorable distribution for ν is a Gaussian distribution. Hence, the two seemingly different problems (with different loss functions -- square-difference distortion and mutual information) admit the same saddle-point solution in the presence of an independent jammer strategy. [This equivalence can in fact be verified directly by making use of some inequalities of Shannon [8] on mutual information.] But, this equivalence does not directly extend to the communication system considered in this paper, and derivation of the saddle-point solution of the communication system of Fig. 1 when the loss function is taken as the mutual information between u and z remains today as a challenging problem.

There exist quite a few results in the literature on worst case designs, wherein the Gaussian distribution has been proven to be the least-favorable distribution (such as the cases of entropy maximization [9], Fisher-information minimization [10], or minimax estimation problems [11], [12]), and the present paper adds to this list a new class of problems not considered heretofore. We should note, however, that if the input sequence in Fig. 1 is vector-valued and/or the number of channels is more than one, the saddle-point solution will no longer be linear-Gaussian (i.e., the solution of this paper does not carry over to the vector case), since the counterpart of the Gaussian test channel does not admit a simple linear coding scheme in the vector case [13].

IV. APPENDIX

In this appendix, we provide proofs for the two Lemmas which were used in the proof of Theorem 1 in section II.

Lemma 1.

$$p = -\alpha^2 + [\alpha(c + \alpha\sigma)/(c^2 + \xi_1 + \sigma)] > 0, \text{ in } \underline{R2} \cap \underline{R3},$$

where α is defined by (14).

Proof. Through straightforward substitution from (14) and (8a), and some manipulations,

$$\begin{aligned} p &= \frac{\alpha^2}{c^2 + \xi_1 + \sigma}\left[\frac{c}{\alpha} + \sigma - c^2 - \xi_1 - \sigma\right] \\ &= \frac{\alpha^2}{c^2 + \xi_1 + \sigma}\left[(1 + \lambda)(c^2 + \xi_1) + \frac{\lambda^2\sigma}{1 + \lambda} + \frac{\xi_2}{1 + \lambda} - c^2 - \xi_1\right] \\ &= \frac{\alpha^2}{(1 + \lambda)(c^2 + \xi_1 + \sigma)}\left[\lambda^2(c^2 + \xi_1 + \sigma) + \lambda(c^2 + \xi_1) + \xi_2\right] \\ &= \frac{\alpha^2}{(1 + \lambda)(c^2 + \xi_1 + \sigma)}\left[k^2 - \frac{k(c^2 + \xi_1)}{(c^2 + \xi_1 + \sigma)^{\frac{1}{2}}} + \xi_2\right] \\ &> 0 \end{aligned}$$

since $1 + \lambda > 0$ in R2, and the last multiplicative term is positive in R3. □

Lemma 2.

$t \in [0,1]$ in $\underline{R2} \cap \underline{R4}$, where t is defined by (8b).

Proof. Starting with the inequality that determines R4,

$$k^2 + \xi_2 \leq \frac{(c^2 + \xi_1)}{(c^2 + \xi_1 + \sigma)^{\frac{1}{2}}} \; k$$

$$\longleftrightarrow \qquad (k^2 + \xi_2)(c^2 + \xi_1 + \sigma)^{\frac{1}{2}} \leq (c^2 + \xi_1)\; k \quad ,$$

and squaring both sides

$$(k^2 + \xi_2)^2(c^2 + \xi_1 + \sigma) \leq (c^2 + \xi_1)^2 k^2$$

we arrive at

$$\frac{(k^2 + \xi_2)^2(c^2 + \xi_1 + \sigma)}{(c^2 + \xi_1)^2 k^2} \leq 1 \quad .$$

Since this latter expression is equal to $1 - t$ [from (8b)], and it is also positive, the desired property follows. $\square$

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MINIMAX FILTERING PROBLEMS FOR OBSERVED POISSON PROCESSES
WITH UNCERTAIN RATE FUNCTIONS*

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Abstract

This paper treats the following decision problems for continuous-time systems with discontinuous observations (i.e., for systems observed through point processes):

I. Robust matched filtering.
II. Robust Wiener filtering
III. Minimax state estimation for systems with noise uncertainty.

In each case there is assumed to be some degree of uncertainty in the rate function of an observed Poisson process, and a corresponding minimax design philosophy is adopted. In Problem I we assume that the rate of the observation process is a deterministic function of time, and in Problems II and III we assume that these rates are wide-sense-stationary stochastic processes. General solutions to the three problems are considered in terms of least-favorable rate functions or processes, and several useful models of uncertainty are discussed in this context.

Introduction

The purpose of this paper is to formulate and solve three minimax decision problems for continuous -time systems observed through point processes. In particular, for Poisson observations with uncertain rates, we consider the problems of robust matched filtering, robust Wiener filtering, and minimax state estimation. All of these problems have been considered in the context of continuous observations. In particular, for continuous observation models, robust matched filtering has been considered in [1-4], robust Wiener filtering in [5-8], and minimax state estimation in [9-10]. Here, we apply the methodology of these earlier works to consider these problems for discontinuous observations.

In Section I, the robust matched filtering problem is considered. We see here that, although the problem formulation and its solution for L_2 uncertainty resemble that of the corresponding problem for continuous signals [3], the discontinuous problem is not directly transformable to the continuous one. Furthermore the special signal structure (the nonnegativity of the rate function) in the discontinuous observations case allows us to consider for the signal (rate) any uncertainty model described by 2-alternating capacities (e.g. contaminated mixtures, total variation neighborhoods, band models, and extended p-point models (see [22])); this cannot be done for the continuous observations case. Therefore, complete reanalysis of this problem is required here. Sections II and III treat the related problems of robust Wiener filtering and minimax state estimation, respectively. In contrast to the matched filtering problem of Section I, it is seen that these two problems can be transformed directly to analogous continuous-time problems (as treated in [7],[8],[10]) and thus are solvable using techniques of previous studies. For these problems we present an approach that treats, in a unified way, some important models for uncertainty in the rates (e.g., band models and contaminated mixtures).

I. ROBUST MATCHED FILTERING

A. System Model

Consider the photodetector depicted in Fig. 1, which is used for fiber and free space optical communication systems (see [11]) and which allows us to emphasize different noise contributions. The output of the photodetector is given by the sum of a filtered Poisson process $i_s(t)$ plus an independent zero-mean thermal noise $i_{th}(t)$. The current $i_s(t)$ can be expressed, for $t \geq 0$, as

$$i_s(t) = \sum_{n=1}^{N_t} eg_n\delta(t-\tau_n) \tag{1.1}$$

where $\{N_t, t \geq 0\}$ is an inhomogeneous counting process such that N_t is the number of photoelectrons generated during $[0,t)$, τ_n is the emission time, and the g_n's are i.d. random variables that, in avalanche photodiodes (APD's), model the number of secondary electrons generated for each primary photoelectron. Here e is the charge of an electron and the APD impulse response is assumed to be $g\delta(t)$ where δ is the Dirac delta (i.e., the photodiode is assumed to be ideal). This is not a major restriction and the results that follow may be modified for the general case in which the impulse response is $gh_s(t)$ with h_s nonimpulsive.

The intensity λ_t of N_t is related to the

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incident optical power by

$$\lambda_t = \frac{\eta}{h\nu} p(t) + \lambda_d = \lambda_s(t) + \lambda_d \qquad (1.2)$$

where η is the quantum efficiency of the APD and $h\nu$ is the energy in a photon, viz., h is Plank's constant and ν is the unmodulated optical carrier frequency. We assume that $p(t)$ is deterministic and thus so is $\lambda_s(t)$ defined in (1.2). The rate λ_d accounts for the constant rate at which spontaneous but extraneous electrons are generated during $[0,t]$. As a consequence $\lambda_s(t)$ is affected by a disturbance called the dark current which is usually negligible provided $p(t)$ is not too small.

The filtered Poisson process at the output of the receiver filter (see Fig. 1) y_t can be written as

$$y_t = \sum_{n=1}^{N_t} e g_n \tilde{h}(t-\tau_n) + \int_0^t i_{th}(\tau)\tilde{h}(t-\tau)d\tau . \qquad (1.3)$$

In [12, pp. 168-179] the characteristic function, the cumulants, and thus the moments of y_t were evaluated. In particular the signal-to-noise ratio at the output of the receiver is given by

$$\mathrm{SNR} = \frac{E^2[y_t \mid \lambda_d = 0, i_{th}(\cdot)=0]}{\mathrm{Var}[y_t]} = \frac{E^2(g)}{E(g^2)}$$

$$\cdot \frac{\left[\int_0^t \tilde{h}(t-\tau)\lambda_s(\tau)d\tau\right]^2}{\int_0^t \tilde{h}^2(t-\tau)\lambda_s(\tau)d\tau + \left[\frac{N_0}{2e^2E(g^2)} + \lambda_d\right]\int_0^t \tilde{h}^2(t-\tau)d\tau} \qquad (1.4)$$

where $N_0/2$ is the two-sided spectral density of the thermal noise process which is assumed to be white Gaussian, and the moments $E(g)$ and $E(g^2)$ of the gain of the APD were evaluated in [13]; for our problem these are given constants.

The assumption in (1.4) about the thermal noise being a white Gaussian process is not restrictive since this is a realistic enough model for our system; notice that in a photon channel the main source of "noise" is the shot noise involved in the counting process N_t. Therefore, in what follows we will be primarily interested in the "signal" process $\lambda_s(\cdot)$.

2. The Matched Filter

As proved in [12, pp. 168-172], as certain parameters tend to prescribed limits, the process $y(t)$ (defined in (1.3)) tends to a Gaussian process on $[0,t)$ with mean the unsquared numerator of (1.4) and variance the denominator of (1.4). Therefore the probability of error reduces to $Q(\mathrm{SNR})$ (where Q is the tail of the standard Gaussian distribution) and SNR becomes a useful performance measure. Thus, maximizing the SNR over all possible filter impulse responses is desirable. The resulting optimal filter is the matched filter.

Let s stand for $\lambda_s(\cdot)$, h for $\tilde{h}(t-\cdot)$, and $n_w = \frac{N_0}{2e^2E(g^2)} + \lambda_d$. Let $\langle\cdot,\cdot\rangle$ be the inner product defined for real a,b as $\langle a,b\rangle = \int_0^t a(\tau)b(\tau)d\tau$. Then $h \in \mathcal{X}$, $s \in \mathcal{X}^+ = \{a : a \in \mathcal{X}, a \geq 0\}$ where $\mathcal{X}$ is $L^2[0,t)$ and SNR of (1.4) may be written as $\mathrm{SNR} = E^2(g)\rho(h;s,n_w)/E(g^2)$ where

$$\rho(h;s,n) = \langle h,s\rangle^2/\langle h,(s+n)h\rangle \qquad (1.5)$$

for any $n \in \tilde{\mathcal{X}}$, the space of bounded, linear, positive operators mapping $\mathcal{X}$ to itself. For the continuous-time case that we consider in this paper, n is the autocorrelation function of the zero-mean Gaussian thermal noise process and it is generally of the form $K_c(\tau,\sigma) + \lambda_d\delta(\tau-\sigma)$. In this general case $s+n$ of (1.5) must be interpreted as $s(\tau)\delta(\tau-\sigma) + n(\tau,\sigma)$. If in particular $n(\tau,\sigma)$ is of the form $n(\tau)\delta(\tau-\sigma)$ (i.e. the noise process is timewise uncorrelated) then $s+n$ of (1.5) is just $s(\tau)+n(\tau)$; the white noise case that we treated first, above, is a special case of this for $n(\tau) = n_w$ for all τ.

The matched filter problem for fixed s and n is given by (compare with [3])

$$\max_{h \in \mathcal{X}} \rho(h;s,n) \qquad (1.6)$$

The solution to this problem for fixed s is given by

Property 1 (Matched Filter):

$$\max_{h \in \mathcal{X}} \rho(h;s,n) = \rho((s+n)^{-1}s;s,n) = \langle s,(s+n)^{-1}s\rangle \qquad (1.7)$$

where $(s+n)^{-1}$ is the inverse mapping in $\tilde{\mathcal{X}}$ corresponding to $s+n$ (recall $s \geq 0$ and n is a positive operator).

Proof. Follows straightforwardly from the Schwarz inequality.

We are going to need the following two properties of the functional ρ.

Property 2: For fixed $h \in \mathcal{X}$, $\rho(h;s,n)$ is convex in $(s,n) \in \mathcal{X}^+ \times \tilde{\mathcal{X}}$.

Proof. The proof is similar to that of Property 2 of [3].

Property 3: The functional $\max_{h \in \mathcal{X}} \rho(h;s,n) = \langle s,(s+n)^{-1}s\rangle$ is convex in (s,n) on $\mathcal{X}^+ \times \tilde{\mathcal{X}}$.

Proof. Since $\langle s,(s+n)^{-1}s\rangle = \rho((s+n)^{-1}s;s,n)$ Property 3 follows from Property 2.

Remark 1. In comparing $\langle s,(s+n)^{-1}s\rangle$ of (1.7) with $\langle s,n^{-1}s\rangle$ of [3] notice that the discontinuous observations case is equivalent to the continuous observations case with autocorrelation function $n(\tau,\sigma) + s(\tau)\delta(\tau-\sigma)$ (i.e. the useful signal also plays the role of additive uncorrelated noise).

Remark 2. Let us assume that $\lambda_d = 0$ (no "dark" current is present) and that no thermal noise disturbs the system of Fig. 1. Then via (1.4),

(1.3) reduces to

$$\rho(h;s,0) = (h,s)^2/(h,sh)$$

and (1.7) to

$$\max_{h\in\mathcal{H}} \rho(h;s,0) = \rho(1,s,0) = (s,1)$$

that is the optimal filter is that one with impulse response identically 1; in other words the matched filter is a pure integrator in this case.

2. The Robust Matched Filter for Uncertain Signal Structure

Equation (1.7) indicates that for known s,n the matched filter is given by $h = (s+n)^{-1}s$. Suppose now, that s and n are only known to be members of the classes $\mathcal{S} \subset \mathcal{H}^+$ and $\mathcal{N} \subset \bar{\mathcal{H}}$. We would like to find a filter whose performance does not deteriorate drastically over $\mathcal{S}\times\mathcal{N}$. Then as in [3] we say that $h_R \in \mathcal{H}$ is robust over $\mathcal{S}\times\mathcal{N}$ if

$$\inf_{(s,n)\in\mathcal{S}\times\mathcal{N}} \rho(h_R;s,n) = \max_{h\in\mathcal{H}}\{\inf_{(s,n)\in\mathcal{S}\times\mathcal{N}} \rho(h;s,n)\} \quad (1.8)$$

We note that $(h_L;s_L,n_L) \in \mathcal{H}\times\mathcal{H}^+\times\bar{\mathcal{H}}$ is a saddle point solution to the game of (1.8) if

$$\inf_{(s,n)\in\mathcal{S}\times\mathcal{N}} \rho(h_L;s,n) \geq \rho(h_L;s_L,n_L) = \max_{h\in\mathcal{H}} \rho(h;s_L,n_L). \quad (1.9)$$

Concerning such a solution we have the following result.

Lemma 1. Suppose $\mathcal{S}$ and $\mathcal{N}$ are convex, $(s_L,n_L)\in\mathcal{S}\times\mathcal{N}$, and $h_L = (s_L+n_L)^{-1}s_L$. Then $(h_L;s_L,n_L)$ is a saddle point for eq. (1.8) iff the following inequality holds for all $(s,n)\in\mathcal{S}\times\mathcal{N}$:

$$2(h_L,s) - (h_L,s_L) \geq (h_L,(s+n)h_L) \quad (1.10)$$

Proof. For $\alpha\in[0,1]$ and $(s,n)\in\mathcal{S}\times\mathcal{N}$ define $\sigma = (1-\alpha)s_L + \alpha s$, $\nu = (1-\alpha)n_L + \alpha n$ and

$$K(\alpha;s,n) = \rho(h_L;\sigma,\nu) .$$

Then, since $\mathcal{S}$ and $\mathcal{N}$ are convex $(h_L;s_L,n_L)$ is a saddle point for eq. (1.9) iff

$$K(\alpha;s,n) \geq K(0;s,n)$$

for all $\alpha\in[0,1]$ and $s\in\mathcal{S}$. Since σ and ν are linear functions of α and $K(\alpha;s,n)$ is through its definition and from Property 2 a convex function of (s,n) it follows that $K(\alpha;s,n)$ is convex in α for each $(s,n)\in\mathcal{S}\times\mathcal{N}$. Thus $K(\alpha;s,n)\geq K(0;s,n)$ holds iff

$$[\partial K(\alpha;s,n)/\partial\alpha]|_{\alpha=0} \geq 0$$

for all $(s,n)\in\mathcal{S}\times\mathcal{N}$. On differentiating we have

$$[\partial K(\alpha;s,n)/\partial\alpha]|_{\alpha=0} = 2(s,h_L) - (s_L,h_L) - (h_L,(s+n)h_L)$$

and Lemma 1 follows.

Now, we define a pair (s_L,n_L) to be least favorable for $\mathcal{S}\times\mathcal{N}$ if

$$(s_L,(s_L+n_L)^{-1}s_L) = \min_{(s,n)\in\mathcal{S}\times\mathcal{N}} (s,(s+n)^{-1}s) . \quad (1.11)$$

It follows easily from Eqs. (1.7), (1.9) and (1.11) that $(s_L,n_L)\in\mathcal{S}\times\mathcal{N}$ is least favorable for $\mathcal{S}$ and $\mathcal{N}$ iff $(h_L;s_L,n_L)$ where $h_L = (s_L+n_L)^{-1}s_L$ is a saddle point solution to (1.8). We also have

Lemma 2: Suppose $\mathcal{S}$ and $\mathcal{N}$ are convex, $(s_L,n_L)\in\mathcal{S}\times\mathcal{N}$ and $h_L = (s_L+n_L)^{-1}s_L$. Furthermore suppose that $(s_L,(\sigma+\nu)^{-1}s_L)$ is right continuous in α at $\alpha = 0$ for each $(s,n)\in\mathcal{S}\times\mathcal{N}$ where $\nu = (1-\alpha)n_L + \alpha n$ and $\sigma = (1-\alpha)s_L + \alpha s$. Then $(s_L,n_L)\in\mathcal{S}\times\mathcal{N}$ is least favorable for $\mathcal{S}$ and $\mathcal{N}$ iff (1.10) holds for all $(s,n)\in\mathcal{S}\times\mathcal{N}$.

Proof. The proof is similar to the proof of Lemma 1 and follows step by step the proof of Lemma 2 and the Appendix of [3].

Lemmas 1 and 2 imply that, within a mild continuity requirement the triple $(h_L;s_L,n_L)$ with $(s_L,n_L)\in\mathcal{S}\times\mathcal{N}$ and $h_L = (s_L+n_L)^{-1}s_L$ gives the desired solution to our problem iff (1.10) is satisfied.

In the sequel we find the robust matched filter h_R for classes $\mathcal{S}$ of the signal described by L_2 uncertainty or uncertainty described by Choquet capacities. Although we could consider uncertainty in the noise (or in the signal and the noise simultaneously), we restrict attention to signal uncertainty only, since the thermal noise can be very realistically modeled as white Gaussian for most applications. Therefore it is not so restrictive to assume that we deal with nonnegative autocorrelation functions. Also recall that the signal s which represents a rate function is always nonnegative. Under these assumptions the continuity requirement of Lemma 2 is satisfied.

C.1 L_2 uncertainty

Let $\mathcal{N} = \{n_0\}$ satisfying the assumptions above and assume that the signal quantity is known to be in the class $\mathcal{S}_1 \subset \mathcal{H}^+$ defined by

$$\mathcal{S}_1 = \{s\in\mathcal{H}^+ | \|s-s_0\|^2 \leq \Delta\} \quad (1.12)$$

where s_0 can be thought of as a known nominal signal and $\Delta > 0$ as a degree of uncertainty in the nominal model.

Define

$$s_L = (s_L + n_0)h_R = s_0 - \sigma_0 h_R(2-h_R) \quad (1.13a)$$

$$\sigma_0^2 \|h_R(2-h_R)\|^2 = \Delta \quad (1.13b)$$

where h_R satisfies the equation

$$n_0 h_R = (1-h_R)[s_0 - \sigma_0 h_R(2-h_R)] . \quad (1.13c)$$

Then we have that if a solution to (1.13c) and (1.13b) exists

Theorem 1: The triple $(h_R;s_L,n_0)$ is a saddle point

solution to eq. (1.9) for $\mathcal{S}_1$ and $\{n_0\}$.

<u>Proof</u>. Since $s_L \in \mathcal{S}_1$ and $h_R = (s_L + n_0)^{-1} s_L$ it is sufficient via Lemma 1 to show that (1.10) with $h_R = h_L$ holds for all $s \in \mathcal{S}_1$. It is easily seen that, since $n_0 h_R = (1 - h_R) s_L$, eq. (1.10) becomes

$$\langle h_R(2 - h_R), s - s_L \rangle \ge 0$$

and then from (1.13a) and (1.13b)

$$\langle h_R(2 - h_R), s - s_0 \rangle - \frac{\epsilon}{\sigma_0} \ge 0.$$

But from the Schwarz inequality, (1.13b) and (1.12)

$$|\langle h_R(2-h_R), s-s_0\rangle| \le \|h_R(2-h_R)\| \cdot \|s-s_0\| \le \frac{\sqrt{A}}{\sigma_0}\sqrt{\delta} = \frac{\epsilon}{\sigma_0}$$

and the theorem follows.

Equation (1.13c), which gives the robust matched filter for our problem, can be solved iteratively together with (1.13b). Solutions to related equations for the continuous observations case have been treated in [4].

<u>1.2 Uncertainty within capacity classes</u>

In this section we consider uncorrelated thermal noise processes with autocorrelation functions of the form $n_0(\tau)\delta(\tau - \sigma)$ where $n_0(\tau) > 0$ a.e. and let us define

$$\int_0^T n_0(\tau)\,d\tau = E_N. \qquad (1.14)$$

The important case of white noise is included in the class above ($n_0(\tau) = N_0/2$ and $E_N = N_0 T/2$ for white noise with two-sided spectral density $N_0/2$). On the nonnegative signal s ($s(\tau) = \lambda_s(\tau)$) we impose the constraint

$$\int_0^T s(\tau)\,d\tau = w_s T = E_s. \qquad (1.15)$$

Equation (1.15) is an average power constraint (w_s denotes average power, E_s denotes average energy). Note that in most optical communications systems the rate function is proportional to the transmitted power (see for example eq. (1.2)).

Before characterizing the uncertainty class for the signal we need some definitions. Let Ω denote $[0,T]$ and $\mathcal{A}$ denote the Borel σ-algebra on Ω. Then a finite set function v on $\mathcal{A}$ is a 2-alternating capacity [18] on $(\Omega,\mathcal{A})$ if it is increasing, continuous from below, continuous from above for closed sets, and if it satisfies $v(\emptyset) = 0$ and $v(A \cup B) + v(A \cap B) \le v(A) + v(B)$ for all $A,B \in \mathcal{A}$. Obviously any finite measure is a 2-alternating capacity.

For any pair (v_0, v_1) of 2-alternating capacities on $(\Omega,\mathcal{A})$ there exists a Radon-Nikodyn derivative dv_1/dv_0, introduced in [19], with the defining property that for each $x \in [0,\infty)$ $d_x(\{dv_1/dv_0 > x\}) = \inf_{A \in \mathcal{A}} d_x(A)$ where $d_x(A) \triangleq (1+x)^{-1}[xv_0(A) + v_1(A^c)]$.

Define the finite measures m_N and m_s so that they are <u>absolutely continuous</u> with respect to the Lebesque measure μ on Ω with derivatives n_0 and s, respectively. (This restriction is only for the purpose of preserving notational uniformity and can be relaxed.) Consider the sets

$$\mathcal{M}_N = \{v_N\} \qquad (1.16)$$

$$\mathcal{M}_s = \{m_s \mid m_s(A) \le v_s(A) \text{ for all } A \in \mathcal{A}, \text{ and } m_s(\Omega) = v_s(\Omega) = E_s\} \qquad (1.17)$$

where $v_N(A) = m_N(A)$ for all $A \in \mathcal{A}$, $v_N(\Omega) = E_N$, and v_s is a 2-alternating capacity. Note that $\mathcal{M}_s$ is weakly compact [19] and convex. The following result is a subcase of Theorem 4.1 of [19].

<u>Lemma 3</u>: Let v_N and v_s be as defined in (1.16)-(1.17), and π_0 be a version of dv_s/dv_N. Then there exists measure $m_s^L \in \mathcal{M}_s$ such that $\pi_0 \in dm_s^L/dv_N$ and

$$m_s^L(\{\pi_0 < x\}) = v_s(\{\pi_0 < x\}) \qquad (1.18)$$

for all $x \in [0,\infty)$. Consequently π_0 becomes stochastically smaller under m_s^L over $\mathcal{M}_s$.

Let s_L be the derivative of m_s^L with respect to the Lebesgue measure on Ω. Next, we prove the main theorem of this section.

<u>Theorem 2</u>. For m_s^L as defined by Lemma 3 let $\pi_0 \in dm_s^L/dv_N$, in particular $\pi_0 = s_L/n_0$, and define $h_L = \pi_0(1+\pi_0)^{-1}$. Then $(h_L; s_L, n_0)$ is a saddle-point solution for the game of (1.8) where the classes $\mathcal{S}$ and $\mathcal{N}$ are defined by $\mathcal{S} = \{s \mid m_s \in \mathcal{M}_s$ and $s = dm_s/d\mu\}$, $\mathcal{N} = \{n_0\}$.

<u>Proof</u>. To prove that $(h_L; s_L, n_0)$ is a saddle-point for (1.8), it suffices to show that (1.10) is true (see Lemma 1). In this case (1.10) reduces to

$$\langle s - s_L, 2h_L - h_L^2 \rangle \ge 0$$

or

$$\int_\Omega \phi(h_L)\,s\,d\mu \ge \int_\Omega \phi(h_L)\,s_L\,d\mu$$

where $\phi(h_L) = 2h_L - h_L^2$ and $h_L = (s_L + n_0)^{-1}s_L = \pi_0(1+\pi_0)^{-1}$. Equivalently we can write

$$\int_\Omega \rho(\pi_0)\,dm_s \ge \int_\Omega \rho(\pi_0)\,dm_s^L$$

where $\rho(\pi_0) = 1 - (1+\pi_0)^{-2}$ is an increasing function of π_0. However, Lemma 3 implies that π_0 becomes stochastically smaller under m_s^L than any other $m_s \in \mathcal{M}_s$. This completes the proof of Theorem 2.

The rate function s_L can be thought as a <u>least-favorable</u> signal for minimax matched filtering. We can actually prove the following property.

<u>Theorem 3</u>. The measure $m_s^* = \pi_s$ satisfies the conclusion of Lemma 3 iff $s_* = dm_s^*/d\mu$ minimizes

$$\min_{\pi\in\tilde{\Pi}} D(h;s,\pi_0) = \int s\,(s+\pi_0)^{-1}\pi_0\,d\mu = \int_{\Omega} s^2\,(s+\pi_0)\,d\mu \quad (1.19)$$

over all $s \in \mathscr{S}$.

<u>Proof</u>. Define $f = dm_s/d(m_s+m_0) = \pi_0(s+\pi_0)^{-1}$. Then

$$\min_{\pi\in\tilde{\Pi}} D(h;s,\pi_0) = \int_{\Omega} (1-f)^2\,d(m_s+m_0).$$

Since $C(x) = (1-x)^2$ is convex and twice differentiable on $[0,1]$, Theorem 3 follows from Theorem 5.1 of [19].

Recall that $\Omega = [0,t]$ is compact for any finite decision time interval t. Therefore all the important classes of signal uncertainty are capacities (see [22]). The least-favorable signal s_* is then related by $s_* = E_s dQ_s/d\mu$ to the least-favorable distribution Q_s of the equivalent robust hypothesis testing problem of

$\mathscr{P}_s = [P_s: P_s = m_s\ E_s(m_s \in \pi_s)]$ versus $\mathscr{P}_N = [m_N\ E_N]$ ($\mathscr{P}_N = [\mu,t]$ in the white noise case). The least-favorable distributions have been found for the ε-contaminated, total-variation neighborhoods models in [20]; for the band model in [21]; and for the extended p-point in [22].

II. ROBUST WIENER FILTERING

Suppose the triple $(\Omega, \mathscr{F}_{t,t\geq 0}, P)$ consists of a sample space Ω, an increasing family of σ-fields $\mathscr{F}_{t,t\geq 0}$ and a probability measure P. Let N_t be an $\mathscr{F}_t$-adapted point process with jumps of size 0 and 1 and compensator $\int_0^t \lambda_s ds$ (see [14]). Consider the problem of <u>linear MMSE estimation</u> of λ_t from the observations N_t given by

$$dN_t = \lambda_t dt + dm_t, \quad t \geq 0 \quad (2.1)$$

where m_t is an $\mathscr{F}_t$ martingale. Note that $\{\lambda_t; t\geq 0\}$ thus defined is the rate of the observations.

The solution to this estimation problem is known [12, pp. 317-320] to be

$$\hat{\lambda}_t = \int_0^t h_0(t,u)\,dN_u \quad (2.2)$$

where $h_0(t,u)$ is the solution to

$$K_\lambda(t,s) = \int_0^t h_0(t,u)[K_\lambda(u,s) + \bar{\lambda}_u \delta(u-s)]\,du, \quad (2.3)$$

$\bar{\lambda}_u = E(\lambda_u)$ and $K_\lambda(t,s) = E(\lambda_t\lambda_s)$ (λ_t is assumed to be nonnegative).

The equation corresponding to (2.3) for continuous observations would be [15, pp. 198-201] the same as (2.3) but with $\bar{\lambda}_u\delta(u-s)$ replaced by $K_n(u,s)$, the autocorrelation of the noise process. Similarly the MSE (mean square error) for any linear estimate $\int_0^t h(t,u)dN_u$ of λ_t is given straightforwardly by

$$E[(\lambda_t - \int_0^t h(t,u)dN_u)^2] = K_\lambda(t,t)$$

$$+ \int_0^t\int_0^t h(t,u)[K_\lambda(u,v) + \bar{\lambda}_u\delta(u-v)]h(t,v)\,du\,dv$$

$$- 2\int_0^t h(t,u)K_\lambda(t,u)\,du \quad (2.4)$$

and it turns out that the corresponding expression for continuous observations would be the same with $\bar{\lambda}_u\delta(u,v)$ replaced by $K_n(u,v)$. Therefore (2.1) can be written in the equivalent form (equivalent for linear filter design, that is)

$$dr_t = \lambda_t dt + (\bar{\lambda}_t)^{\frac{1}{2}} dw_t, \quad t \geq 0 \quad (2.5)$$

where r_t is the continuous observations process and w_t is a standard (zero drift, variance t) Wiener process. This result appears in [12, p. 320].

For λ_t being wide-sense stationary, we have $\bar{\lambda}_t = \bar{\lambda}, K_\lambda(t,s) = K_\lambda(t-s)$ and (2.5) reduces to

$$dr_t = \lambda_t dt + (\bar{\lambda})^{\frac{1}{2}} dw_t = \lambda_t dt + d\tilde{w}_t \quad (2.6)$$

where $\dot{\tilde{w}}_t$ is white Gaussian noise with double-sided spectral density $\bar{\lambda}$.

Therefore the <u>noncausal</u> Wiener filtering error is given by [15, p. 496] where $\bar{\lambda}$ replaces $N_0/2$. In particular we have

$$e^*(S,\lambda_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S(\omega)\bar{\lambda}}{S(\omega)+\bar{\lambda}}\,d\omega, \quad (2.7)$$

and the <u>causal</u> Wiener filtering error [15, p. 501], [16] is given by

$$e^*(S,\lambda_0) = \frac{\bar{\lambda}}{2\pi}\int_{-\infty}^{\infty} \ln[1+\frac{S(\omega)}{\bar{\lambda}}]\,d\omega \quad (2.8)$$

where $S(\omega)$ is the spectral density of λ_t.

If the process λ_t does not have spectral density $S(\omega)$ but has a spectral measure m_s then (2.7) becomes

$$e^*(m_s,m_N) = \frac{1}{2\pi}\int \frac{dm_s\,dm_N}{dm_s + dm_N} \quad (2.9)$$

where $dm_N = \bar{\lambda}d\omega$. However as shown in [17], (2.8) remains the same where now $S(\omega)$ is the derivative of the absolutely continuous part of m_s with respect to the Lebesgue measure.

Note that, in all of the above discussion, it is assumed that the second-order characteristics of the rate process $\{\lambda_t;\ t\geq 0\}$ are known. However, since these characteristics are rarely known exactly, we wish to consider the filter design problem for the case in which the rate spectrum S

is known only to be within a convex class $\mathscr{S}$ of spectra. The robust Wiener filtering problem then becomes: find the least favorable f over $\mathscr{S}$, i.e. find $f \in \mathscr{S}$ solving

$$\min_{f \in \mathscr{S}} e(f) \quad \text{or} \quad \min_{f \in \mathscr{S}} \bar{e}(f),$$

and then design the optimum filter for this spectrum; this solution is then (via [5]) a robust (minimax) MSE filter. Therefore there is a direct analogy with the continuous observations case when the noise is restricted to be white. The spectral measure of the noise in this case is a Lebesgue measure over $(-\infty,\infty)$, and thus the least-favorable spectral measure for the signal can be obtained for capacity classes via [23].

If the spectral density is bandlimited then all the uncertainty classes mentioned in Section II.3.2 are capacities; otherwise only the band model gives a capacity. However, in the noncausal filtering case and for an ϵ-contaminated signal the least-favorable density was found in [3] following a direct approach.

III. MINIMAX STATE ESTIMATION FOR LINEAR SYSTEMS WITH NOISE UNCERTAINTY

In this section we take $(\Omega, \mathscr{F}_t, t \ge 0, P)$ to be defined as above and N_t to be an $\mathscr{F}_t$-adapted point process with jumps 0 or 1 and compensator $\int_0^t \lambda_s ds$.

Consider the linear stochastic system described by

state eq. $\quad dX_t = A_t X_t dt + B_t dv_t, \; t \ge 0, \; X_{t=0} = X_0 \qquad (3.1)$

observations eq. $\quad dN_t = (\bar{\lambda}_t + c_t' X_t) dt + dm_t, \; t \ge 0 \qquad (3.2)$

where A_t and B_t are matrices (B_t positive definite), c_t is a vector, v_t is a vector standard Wiener process independent of X_0 (Σ_0, the covariance of X_0, is given, $E(X_0) = 0$), and m_t is an $\mathscr{F}_t$-martingale. The processes v_t and m_t are assumed to be independent.

It is known in [12, p. 23] that the Kalman filtering equations for the system described in (3.1)-(3.2) are

$$d\hat{X}_t = A_t \hat{X}_t dt + \Sigma_t c_t (\bar{\lambda}_t)^{-1} [dN_t - \bar{\lambda}_t dt - c_t' \hat{X}_t dt] \qquad (3.3a)$$

$$\hat{X}_0 = 0$$

$$\frac{d\Sigma_t}{dt} = A_t \Sigma_t + \Sigma_t A_t' + B_t B_t' - \Sigma_t c_t (\bar{\lambda}_t)^{-1} c_t' \Sigma_t \qquad (3.3b)$$

$$\Sigma_{t=0} = \Sigma_0$$

$$\hat{\lambda}_t = \bar{\lambda}_t + c_t' \hat{X}_t \qquad (3.3c)$$

which are the same with those of a continuous-observations system with $\bar{\lambda}_t$ as the covariance of a Wiener process type noise. Also, the performance of any linear filter is predicted by the continuous-observations analog implied by (3.3). In other words an equivalent continuous-observations model to (3.2) is

$$dr_t = (\bar{\lambda}_t + c_t' X_t) dt + (\bar{\lambda}_t)^{1/2} dw_t \qquad (3.4)$$

where w_t is a standard Wiener process.

If, in addition, in the above equations ((3.1)-(3.2)) we assume that $A_t = A$, $B_t = B$, and $c_t = c$ (i.e., we have a time-invariant system), then X_t and λ_t are wide-sense stationary and $\bar{\lambda}_t = \bar{\lambda}$ as in Section II. In this case, we can use the model of (3.4) to apply some of the results of [10] (again [23] and [8] will be helpful as for Section II) directly to the problem of minimax mean-square-error estimation of the state in (3.1) for situations in which there is uncertainty concerning the noise structure (i.e., uncertainty in B or in the covariance matrix of $\{v_t; t \ge 0\}$). In particular, such problems are solved by the Kalman filters corresponding to certain least-favorable noise models that are characterized in [10].

<u>Remark 1</u>. The model of the rate as $\lambda_t = \bar{\lambda} + c' X_t$ can be interpreted as $\bar{\lambda}$ being the carrier and $c' X_t$ containing the information. Thus the useful information (assume $E(X_t) = 0$) is in the variation of X_t, while the power of the "noise" depends on $\bar{\lambda}$.

<u>Remark 2</u>. If (3.1) (being $dX_t = AX_t dt + B dv_t$ in the stationary case) allows X_t to take on both negative and positive values then $\lambda_t = \bar{\lambda} + c' X_t$ becomes negative for some range of X_t and t. However in situations of practical interest this happens with low probability. The exact analysis of this situation remains an open problem (see [24] for a discussion of this problem).

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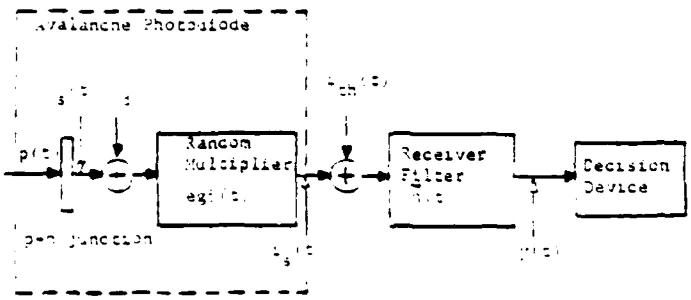

Figure 1. Photodetector Model

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MINIMAX CONTROL OF LINEAR STOCHASTIC SYSTEMS WITH NOISE UNCERTAINTY

by

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Abstract

The problem of linear-quadratic-Gaussian control of multivariable linear stochastic systems with uncertain second-order statistical properties is considered. Uncertainty is modeled by allowing process and observation noise spectral density matrices to vary arbitrarily within given classes, and a minimax control formulation is applied to the quadratic objective functional. General theorems proving the existence and characterization of saddle-point solutions to this problem are presented, and the relationship of these results to earlier results on minimax state estimation are discussed. To illustrate the analytical results, the specific example of regulating a double-integrator plant is treated in detail.

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1. Introduction

The design of optimum decision and control procedures for a linear stochastic system requires an accurate description of the statistical behavior of the system. However, because of nonideal effects such as nonstationarity, nonlinearity, and other modeling inaccuracies, there is always a degree of uncertainty in such statistical descriptions. A useful approach to design in the presence of small modeling inaccuracies is to use a game-theoretic formulation in which one optimizes worst-case performance, and this approach has been applied successfully to many aspects of decision and control system design (see, for example, Huber [1] and Mintz [2]). In a recent paper [3], two of the authors have applied this approach in considering the problem of designing linear minimax-mean-square-error state estimators for linear systems observed in and driven by noise processes with uncertain second-order statistics. In particular, it is shown in [3] that, for two general formulations, such estimators can often be designed by designing linear minimum-mean-square-error filters for least-favorable pairs of noise spectra or covariance matrices. Related minimax state estimation results are found in a paper by Morris [4].

In this paper, we consider the analogous problem of minimax linear-quadratic-Gaussian control (LQG) of systems with uncertain second-order statistics. In particular, we consider the control of linear multivariable systems with white Gaussian process and observation noises with uncertain spectral density matrices. It is shown here that, within mild conditions, this problem can be solved by designing an optimal control for a least-favorable model, although the model which is least-favorable for control may not be the same as that which is least-favorable for state estimation

for the same type of noise uncertainty. However, it is also shown that, for uncertainty in eitherthe process or observation noise only, a given minimax linear-quadratic-Gaussian control problem does have the same least-favorable model as does a particular minimax state estimation problem with a weighted-mean-square-error criterion. Thus, as might be expected, a limited duality exists between these two problems. Another phenomenon which is shown to be associated with minimax control is that the separation principle which separates the problems of optimal control and optimal state estimation is not necessarily valid for minimax control and minimax state estimation. In particular, it is shown that, although the minimax control law is independent of the minimax state estimator, the reverse is not true. Several other aspects of this problem are also considered in this paper.

In Section 2, the specific problem formulation to be considered is given, and several relevant properties of optimal LQG control are outlined. The general minimax problem is treated in Section 3, and results giving conditions for the existence and characterization of saddle-point solutions are derived. Section 4 includes a discussion of several interesting properties associated with the general minimax results of Section 3, and the specific example of controlling a double-integrator plant with uncertain process noise statistics is considered in detail in Section 5.

2. Preliminaries

Consider the linear time invariant stochastic system

$$\dot{x}_t = Ax_t + Bu_t + \xi_t \tag{1}$$

$$y_t = Cx_t + \theta_t \tag{2}$$

where x_t and ξ_t are in $\mathbb{R}^n$, u_t is in $\mathbb{R}^m$, and y_t and θ_t are in $\mathbb{R}^p$ for each t. The matrices A, B and C are assumed to have compatible dimensions (as required by (1)-(2)) with the pairs (A,B) and (A,C) stabilizable and detectable respectively. The noise processes ξ_t and θ_t are assumed to be zero mean white Gaussian processes with second order statistics

$$\begin{aligned} &E\, \xi_t \theta_s^T = 0 \\ &E \xi_t \xi_s^T = \Xi\, \delta(t-s) \\ &E\, \theta_t \theta_s^T = \Theta\, \delta(t-s) \end{aligned} \tag{3}$$

where δ is the Dirac impulse. It is assumed that $(A,\sqrt{\Xi})$ is stabilizable and that $\Theta > 0$. The objective of the problem is to choose u_t to minimize the time average quadratic cost

$$J = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left(x_t^T Qx_t + u_t^T Ru_t\right) dt \tag{4}$$

where $Q \geq 0$ with $(A,\sqrt{Q})$ detectable and $R > 0$.

When Ξ and Θ are known, the solution to the stochastic regulator problem (1)-(4) is given by the feedback system:

$$u_t = -G\hat{x}_t \tag{5}$$

$$\dot{\hat{x}}_t = A\hat{x}_t + Bu_t + H\left(y_t - C\hat{x}_t\right) \tag{6}$$

where

$$G = R^{-1}B^TK \tag{7}$$

$$A^TK + KA + Q - KBR^{-1}B^TK = 0 \tag{8}$$

$$H = PC^T\Theta^{-1} \tag{9}$$

$$AP + PA^T + \Xi - PC^T\Theta^{-1}CP = 0 \tag{10}$$

The matrices K and P are the unique positive semi-definite stabilizing solutions to (8) and (10) respectively.

As discussed in Section 1, the second order statistics for the processes ξ_t and θ_t are often not known precisely. A common representation of this type of uncertainty is to assume that Ξ and Θ are contained in compact sets $\mathcal{X}$ and $\mathcal{H}$ respectively. The objective is then to choose u_t to minimize the worst possible performance (4) given $(\Xi,\Theta) \in \mathcal{X} \times \mathcal{H}$. We will restrict our consideration to controls generated by causal appropriately measurable[1] functions of the measurement. Denote this class of operators as $\mathcal{L}_s^+$. The problem can then be stated as the minimax problem:

$$\min_{L \in \mathcal{L}_s^+} \max_{(\Xi,\Theta)\in \mathcal{X}\times\mathcal{H}} J(L,\Xi,\Theta) \tag{11}$$

where the dependence of J defined by (1)-(4) on L, Ξ, and Θ has been explicitly noted. Note that the optimal linear feedback law defined by (5)-(10) is a member of $\mathcal{L}_s^+$.

[1]See, for example, Chapter 16 of [7] for the explicit measurability conditions.

3. Existence and Characterization of a Saddlepoint

Two important results concerning solutions to the minimax problem formulated in section 2 are presented in this section. The first result establishes an equivalence between a saddlepoint solution to (11) and an optimal stochastic regulator solution (5)-(10) corresponding to a particular (Ξ,Θ) pair. The second result establishes the existence of a saddlepoint when the sets $\mathcal{X}$ and $\mathcal{N}$ are convex.

To obtain these results, we will need the following well-known theorem (cf. [5]) which establishes the fact that the existence of a saddlepoint is a necessary and sufficient condition for the minimax problem (11) to be equivalent to the corresponding maximin problem

$$\max_{(\Xi,\Theta)\in\mathcal{X}\times\mathcal{N}} \; \min_{L\in\mathcal{L}_s^+} J(L,\Xi,\Theta) \tag{12}$$

Theorem 1: There exists a triplet $(L_o,\Xi_o,\Theta_o) \in \mathcal{L}_s^+ \times \mathcal{X} \times \mathcal{N}$ satisfying the saddlepoint condition

$$J(L_o,\Xi,\Theta) \le J(L_o,\Xi_o,\Theta_o) \le J(L,\Xi_o,\Theta_o) \tag{13}$$

$$\forall\; L \in \mathcal{L}_s^+,\; \Xi \in \mathcal{X},\; \Theta \in \mathcal{N}$$

if and only if the values of (11) and (12) are equal.

We will also require the following lemma which expresses the cost for any Ξ and Θ when the control is generated by (5)-(8) with H being any matrix such that (A-HC) is asymptotically stable.

Lemma 1: Assume that the control u_t is generated by the system (5)-(6) with feedback gain G determined by (7)-(8), and that H is any matrix such that all eigenvalues of (A-HC) have negative real parts. Then the cost

J defined by (1)-(8) is:

$$J = \mathrm{tr}\, \Xi\, K + \mathrm{tr}(\Xi + H\Theta H^T)X \tag{14}$$

where K is given by (8) and X is the unique positive semi-definite solution of

$$(A-HC)^T X + X(A-HC) + G^T RG = 0 \tag{15}$$

<u>Proof</u>: A proof is provided in the Appendix[1].

Theorem 2 provides the desired characterization of a saddlepoint.

<u>Theorem 2</u>: Assume there exists $\Xi_o \in \mathcal{X}$ and $\Theta_o \in \mathcal{N}$ which satisfy

$$\mathrm{tr}\{\Xi\, Y\} \le \mathrm{tr}\{\Xi_o\, Y\} \qquad \forall\, \Xi \in \mathcal{X} \tag{16}$$

$$\mathrm{tr}\{\Theta H_o X H_o^T\} \le \mathrm{tr}\{\Theta_o H_o X H_o^T\} \quad \forall\, \Theta \in \mathcal{N} \tag{17}$$

where H_o is the Kalman filter gain corresponding to Ξ_o and Θ_o (given by (9)-(10)), X is given by (15), Y is the solution to

$$(A-H_oC)^T Y + Y(A-H_oC) + Q + KH_oC + C^T H_o^T K = 0 \tag{18}$$

and G and K are given by (7)-(8). Let L_o be the operator representing the optimal stochastic regulator (5)-(6) corresponding to Ξ_o and Θ_o. Then (L_o, Ξ_o, Θ_o) is a saddlepoint solution to (11).

Conversely, assume that (L_o, Ξ_o, Θ_o) is a saddlepoint for (11). Then L_o is the LQG regulator (5)-(10) and (Ξ_o, Θ_o) satisfy (15)-(18).

[1]This result may also be developed using the results on pp. 185-186 of [8].

Proof: (Sufficiency) Consider the maximin problem (12). Let Ξ_o and Θ_o satisfy (15)-(18) and let L_o be the corresponding optimal stochastic regulator. Let H_o be the Kalman gain for Ξ_o and Θ_o given by (9)-(10). Then, by lemma 1

$$J(L_o,\Xi,\Theta) = tr\{\Xi(X+K)\} + tr\{H\Theta H^T X\} \tag{19}$$

for every $(\Xi,\Theta) \in \mathcal{Z} \times \mathcal{N}$. Adding (15) and (8) gives:

$$(A-H_oC)^T(X+K) + (X+K)(A-H_oC) + Q + KH_oC + C^TH_o^TK = 0 \tag{20}$$

Hence

$$Y = X + K \tag{21}$$

Also, by (16)

$$tr\{\Xi(X+K)\} \le tr\{\Xi_o(X+K)\} \tag{22}$$

Adding (22) and (17), and using (19) gives the lower inequality of (13)

$$J(L_o,\Xi,\Theta) \le J(L_o,\Xi_o,\Theta_o) \qquad \forall\ \Xi \in \mathcal{Z}, \Theta \in \mathcal{N} \tag{23}$$

The upper inequality of the saddlepoint condition (13) follows trivially from the fact that L_o is the optimal stochastic regulator. Thus, (L_o,Ξ_o,Θ_o) is a saddlepoint for (11).

(Necessity) Suppose (L_o,Ξ_o,Θ_o) satisfies (13). The upper inequality of (13) implies that L_o is the optimal stochastic regulator (for which one realization is (5)-(10)). Hence lemma 1 can be used to express the cost. The lower inequality and lemma 1 imply:

$$\mathrm{tr}\{\Xi K\} + \mathrm{tr}\{(\Xi + H_o \Theta H_o^T)X\} \le \mathrm{tr}\{\Xi_o K\} + \mathrm{tr}\{(\Xi_o + H_o \Theta_o H_o^T)X\} \tag{24}$$

for every $\Xi \in \mathcal{X}$ and $\Theta \in \mathcal{T}$. By (21) this can be written as

$$\mathrm{tr}\{(\Xi - \Xi_o)Y\} + \mathrm{tr}\{(\Theta - \Theta_o)H_o^T X H_o\} \le 0 \qquad \forall\ \Xi \in \mathcal{X}, \Theta \in \mathcal{T} \tag{25}$$

In particular, $\Theta = \Theta_o$ gives (16) and $\Xi = \Xi_o$ gives (17).

□

Thus, we see that conditions (15)-(18) are equivalent to the existence of a saddlepoint. If such a saddlepoint exists, then the minimax controller is simply the optimal stochastic regulator designed for the particular (Ξ_o, Θ_o) pair which satisfies (15)-(18). This result can be used to establish the existence of a saddlepoint.

Theorem 3: Assume $\mathcal{X}$ and $\mathcal{T}$ are convex, compact sets such that if $\Xi \in \mathcal{X}$ then $\Xi \ge 0$ and $(A, \sqrt{\Xi})$ is stabilizable and if $\Theta \in \mathcal{T}$ then $\Theta > 0$. Then a saddlepoint solution for the minimax problem (11) exists.

Proof: The proof shows that a solution to the maximin problem (12) exists and satisfies conditions (15)-(18) of Theorem 2.

By Lemma 1, and equations (7)-(10) and (15),

$$\begin{aligned} \min_{L \in \mathcal{L}_s^+} J(L, \Xi, \Theta) &= \mathrm{tr}\,\Xi K + \mathrm{tr}(\Xi + \bar{H} \Theta \bar{H}^T)X \\ &\triangleq M(\Xi, \Theta) \end{aligned} \tag{26}$$

is continuous in Ξ and Θ (with $\bar{H}$ given by (9)-(10) for each Ξ and Θ). Since $\mathcal{X}$ and $\mathcal{T}$ are compact, a solution to (12) exists. Let (L_o, Ξ_o, Θ_o) be such a solution.

Then the Frechet differential of (26) with respect to Ξ and Θ at (Ξ_o,Θ_o) must be nonpositive in every direction into the set $\mathcal{X} \times \mathcal{H}$. The Frechet differential of (26) is given by:

$$\begin{aligned}\delta M(\Xi,\Theta;\Delta\Xi,\Delta\Theta) = \ & \mathrm{tr}\{\Delta\Xi(K+X)\} \\ & + \mathrm{tr}\{\Delta\Theta\bar{H}^T X\bar{H}\} \\ & + \mathrm{tr}\{\Theta(\delta\bar{H}^T X\bar{H} + \bar{H}^T X\delta\bar{H})\} \\ & + \mathrm{tr}\{(\Xi+\bar{H}\,\Theta\,\bar{H}^T)\delta X\}\end{aligned} \tag{27}$$

In (27), $\delta\bar{H}$ and δX represent the Frechet differentials of $\bar{H}$ and X with respect to Ξ and Θ. From (15), δX can be computed as the solution of

$$(A-\bar{H}C)^T\delta X + \delta X(A-\bar{H}C) - C^T\delta\bar{H}^T X - X\delta\bar{H}C = 0 \tag{28}$$

Thus, δX is given by:

$$\delta X = -\int_o^\infty e^{(A-\bar{H}C)^T t}[C^T\delta\bar{H}^T X + X\delta\bar{H}C]e^{(A-\bar{H}C)t}dt \tag{29}$$

Substituting (29) into (27) and using a few trace manipulations gives

$$\begin{aligned}\delta M(\Xi,\Theta;\Delta\Xi,\Delta\Theta) = \ & \mathrm{tr}\{\Delta\Xi(X+K)\} + \mathrm{tr}\{\Delta\Theta\bar{H}^T X\bar{H}\} \\ & - \mathrm{tr}\{\int_o^\infty e^{(A-\bar{H}C)t}(\Xi+\bar{H}\Theta\bar{H}^T)e^{(A-\bar{H}C)^T t}dt[C^T\delta\bar{H}^T X + X\delta\bar{H}C]\} \\ & + \mathrm{tr}\{PC^T\delta\bar{H}^T X + PX\delta\bar{H}C\}\end{aligned} \tag{30}$$

But the integral in the third term of (30) is the solution to (10); i.e., P. Hence

$$\delta M(\Xi,\Theta;\Delta\Xi,\Delta\Theta) = \mathrm{tr}\{\Delta\Xi(X+K)\} + \mathrm{tr}\{\Delta\Theta\,\bar{H}^T X\bar{H}\} \tag{31}$$

Consider an arbitrary point $(\Xi,\Theta) \in \mathcal{X} \times \mathcal{H}$. Since $\mathcal{X}$ and $\mathcal{H}$ are convex, the line segment joining (Ξ_o,Θ_o) and (Ξ,Θ) is in $\mathcal{X}\times\mathcal{H}$ and hence

$$(\Delta\Xi,\Delta\Theta) = (\Xi - \Xi_o, \Theta - \Theta_o) \tag{32}$$

is a direction into $\mathcal{Z} \times \mathcal{T}$. Substituting (32) into (31), requiring (31) to be nonpositive and using (21) gives:

$$\mathrm{tr}\{(\Xi - \Xi_o)Y\} + \mathrm{tr}\{(\Theta - \Theta_o)H_o^T X H_o\} \leq 0 \tag{33}$$

The choice $(\Xi,\Theta) = (\Xi,\Theta_o)$ in (33) gives (16) while the choice $(\Xi,\Theta) = (\Xi_o,\Theta)$ in (33) gives (17). Thus, by Theorem 2, (Ξ_o,Θ_o) is a saddlepoint for (28).

$\square$

This section has provided two major results. First, every saddlepoint solution to the minimax problem formulated in section 2 has been characterized by the conditions of theorem 2. In addition to providing a means of identifying a particular solution, these conditions can be used to characterize the set of possible solutions. This subject will be addressed further in the next section. Theorem 3 provides the second important result by demonstrating the existence of a saddlepoint solution to the minimax problem when the sets $\mathcal{Z}$ and $\mathcal{T}$ are convex and compact.

4. Discussion

There are several interesting observations which can be made concerning the results of the previous section. First we note that, since (6), (9), and (10) give the linear least-squares state estimator for a fixed (Θ,Ξ) pair, the optimal linear regulator problem for fixed (Θ,Ξ) is solved by feeding back optimal state estimates through the gain G (which does not depend on (Θ,Ξ)). Thus, as is well known, there is a separation between the estimator and regulator design problems in the case of fixed (Θ,Ξ). However, it follows from Theorem 2, (16), and (17) that such a separation does not generally exist in the minimax problem. In particular we see from Theorem 2 that, although the feedback gain does not depend on (Θ,Ξ), the state estimates used for minimax control are not generally the minimax-mean-square-error state estimates. This follows because the equations determining the least-favorable pair for control depend directly on the cost matrices Q and R, which of course have no effect on which pair is least-favorable for state estimation (as in [3]).

The above observation also implies that the (Θ,Ξ) pair which is least favorable for control is not necessarily the same as that which is least favorable for state estimation. However, the conditions that Theorem 2 requires for minimax control are similar in structure to conditions required by Theorem 5 of [3] for minimax state estimation. Using the similarity it follows that, for fixed Ξ, the Kalman filter corresponding to (Θ_o,Ξ) where Θ_o is from (17) also solves the minimax state estimation problem

$$\min_{\hat{x}_t} \max_{\Theta\in\mathcal{M}} E\{(x_t - \hat{x}_t)^T G_o^T R G_o (x_t - \hat{x}_t)\} \tag{34}$$

where G_o is the regulator feedback gain from (7). A similar statement

applies if Θ is fixed and Ξ is unknown; however if both Ξ and Θ are unknown, there generally is not a single minimax-mean-square-error state estimation problem which has the same least favorable pair as (5).

A maximal element Ξ_M for the set $\mathcal{X}$ is one which satisfies $\Xi \leq \Xi_M$ for all $\Xi \in \mathcal{X}$, where $\Xi \leq \Xi_M$ means that $(\Xi_M - \Xi)$ is nonnegative definite. It was noted in [3] that if $\mathcal{X}$ or η has a maximal element then that element is least favorable for state estimation. By inspection of (16) and (17) we see that maximal elements, when they exist, are also least favorable for the regulator problem.

<u>Example</u>: Assume that the process noise can be written as

$$\xi_t = D\hat{\xi}_t$$

where $\hat{\xi}_t \in \mathbb{R}^q$ is a zero mean white Gaussian process with

$$E\{\hat{\xi}_t \hat{\xi}_\tau'\} = \hat{\Xi}\,\delta(t-\tau)$$

and where (A,D) is a stabilizable pair. A common method of modeling uncertainty in the second order <u>a priori</u> statistics of system (1)-(2) while preserving the input structure of the process noise is to choose a nominal pair $(\hat{\Xi}_N, \Theta_N)$ and assume that the true $\hat{\Xi}$ and Θ differ from the nominal in norm by no more than some positive constant ϵ. Define

$$\mathcal{X} \triangleq \{D\,\hat{\Xi}\,D^T : \|\hat{\Xi} - \hat{\Xi}_N\| \leq \epsilon\}$$
$$\eta \triangleq \{\Theta : \|\Theta - \Theta_N\| \leq \epsilon\}$$

where $\|\cdot\|$ denotes the norm induced by the Euclidean vector norm on the underlying space ($\mathbb{R}^q$ and $\mathbb{R}^p$ respectively) and where

$$\epsilon > \min\{\lambda_{min}(\hat{\Xi}_N), \lambda_{min}(\Theta)\}.$$

Then each set has a maximal element

$$\Xi_o = D(\hat{\Xi}_N + \gamma I)D^T$$

$$\Theta_o = \Theta_N + \gamma I$$

and, by the above discussion the minimax controller is the LQG controller designed for $\Xi = \Xi_o$ and $\Theta = \Theta_o$. $\blacksquare$

Of course, most uncertainty classes of interest will not contain maximal elements; however, for any compact classes $\mathcal{X}$ and $\mathcal{N}$ the set of possible least-favorable pairs can be reduced to only those on the upper boundaries of $\mathcal{X}$ and $\mathcal{N}$, where the upper boundary [6] of $\mathcal{X}$ is the set of Ξ satisfying $\{F \in \mathcal{X} | F \geq \Xi\} = \{\Xi\}$. (Note that the upper boundary of a set with a maximal element is just that maximal element.) Furthermore, note that the conditions (16) and (17) are satisfied for all (Ξ,Θ) pairs in $\mathcal{X} \times \mathcal{N}$ if they are satisfied for all pairs on the upper boundaries of $\mathcal{X}$ and $\mathcal{N}$. Thus, by Theorem 2 the minimax problem on $\mathcal{X} \times \mathcal{N}$ can be equivalently defined on the upper boundaries of $\mathcal{X}$ and $\mathcal{N}$. Moreover, the requirement that $\mathcal{X}$ and $\mathcal{N}$ be convex can be relaxed to the requirement that their upper boundaries are also the upper boundaries of their respective closed convex hulls. We may summarize these observations in the following theorem:

Theorem 4: Suppose the following conditions hold:

(i) $\mathcal{X}$ and $\mathcal{N}$ contain their respective upper boundaries $\mathcal{B}_{\mathcal{X}}$ and $\mathcal{B}_{\mathcal{N}}$.

(ii) $\mathcal{B}_{\mathcal{X}}$ and $\mathcal{B}_{\mathcal{N}}$ are the upper boundaries of their respective closed convex hulls $co(\mathcal{B}_{\mathcal{X}})$ and $co(\mathcal{B}_{\mathcal{N}})$.

(iii) $\forall\, \Xi \in co(\mathcal{B}_{\mathcal{X}})$, $(A,\sqrt{\Xi})$ is a stabilizable pair.

Then:

$$\min_{L \in \mathcal{L}_s^+} \max_{(\Xi,\Theta)\in\mathcal{X}\times\mathcal{N}} J(L,\Xi,\Theta)$$
$$= \min_{L \in \mathcal{L}_s^+} \max_{(\Xi,\Theta)\in\mathcal{B}_{\mathcal{X}}\times\mathcal{B}_{\mathcal{N}}} J(L,\Xi,\Theta). \tag{35}$$

Note that Theorem 4 implies that the compactness and convexity conditions required for existence in Theorem 3 need only apply on and near the upper boundaries of $\mathcal{X}$ and $\mathcal{Y}$.

5. Example

To illustrate the results of the above sections we consider the following example which corresponds to control of a double-integrator plant:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \Xi = \begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix}$$

$$C = [1 \;\; 0]; \quad \Theta = r^2; \quad R = 1; \text{ and } Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{36}$$

where $r^2 > 0$. Note that any compact set of Θ's has a maximal element in this case given by the maximum value of r^2 in the set. Thus, no significant generality is lost by assuming that r^2 is fixed at sup Θ. For fixed q_1, q_2, and q_3, the matrix P solving (10) is given by

$$P = \begin{bmatrix} r\sqrt{2r\sqrt{q_2} + q_1} & r\sqrt{q_2} \\ r\sqrt{q_2} & \sqrt{q_2(2r\sqrt{q_2} + q_1)} - q_3 \end{bmatrix} \tag{37}$$

[illegible] from [illegible] equals

$$r^{-1}\begin{bmatrix} \sqrt{2r\sqrt{q_2} + q_1} \\ \sqrt{q_2} \end{bmatrix} \tag{38}$$

$$[illegible] \tag{39}$$

[illegible] = diag (γ_1, γ_2) where

$$y_1 = \frac{2\sqrt{2(2r\sqrt{q_2} + q_1)} + r + 2\sqrt{q_2}}{2\sqrt{2r\sqrt{q_2} + q_1}} \tag{40}$$

and

$$y_2 = \frac{r^2 + 6\sqrt{q_2}\, r + 2q_1 + 2(r + \sqrt{q_2})\sqrt{2(2r\sqrt{q_2} + q_1)}}{2\sqrt{q_2}\,\sqrt{2r\sqrt{q_2} + q_1}} \tag{41}$$

A convex uncertainty class for Ξ is equivalent to a convex subset of $\{q \in \mathbf{R}^3 | \; q_1 \geq 0,\ q_2 \geq 0, \text{ and } q_1 q_2 \geq q_3^2\}$. Equation (16) and the fact that Y is diagonal imply that $\Xi_o \in \mathcal{X}$ is least favorable iff.

$$(q_2^{(o)} - q_2)y_2^{(o)} + (q_1^{(o)} - q_1)y_1^{(o)} \geq 0; \quad \forall\, \Xi \in \mathcal{X}, \tag{42}$$

where superscripts 'o' denote quantities corresponding to Ξ_o. It follows from (14), (15), (38), and (42) that the state estimation filter structure, the control gain, and the control cost are all independent of q_3. Thus we can set $q_3 = 0$ without loss of generality. (Note that tr(P) does depend on q_3.) To illustrate the solution to (42), we consider the two uncertainty classes ($q_3 = 0$)

$$\mathcal{X}_1 = \{\Xi | \; q_1 > 0, q_2 > 0, \text{ and } \max\{|q_1 - q_1^{(N)}|, |q_2 - q_2^{(N)}|\} \leq \epsilon\} \tag{43}$$

and

$$\mathcal{X}_2 = \{\Xi | \; q_1 > 0, q_2 > 0, \text{ and } |q_1 - q_1^{(N)}| + |q_2 - q_2^{(N)}| \leq \epsilon\} \tag{44}$$

where

$$\Xi_N = \begin{bmatrix} q_1^{(N)} & 0 \\ 0 & q_2^{(N)} \end{bmatrix}$$

represents a nominal model for the state noise and ε is a fixed degree of uncertainty in the model. Note that $\mathcal{X}_1$ has a maximal element

$$\begin{pmatrix} q_1^{(N)} + \varepsilon & 0 \\ 0 & q_2^{(N)} + \varepsilon \end{pmatrix}$$

which thus yields a minimax design immediately for this case. The class $\mathcal{X}_2$ does not have a maximal element for $\varepsilon > 0$, but Theorem 3 allows us to restrict consideration to the upper boundary of $\mathcal{X}_2$ given by

$$\mathcal{B}_{\mathcal{X}_2} = \{\Xi \in \mathcal{X}_2 | q_1 - q_1^{(N)} + q_2 - q_2^{(N)} = \varepsilon\}. \tag{45}$$

Thus (42) reduces in this case to

$$(q_2^{(o)} - q_2)(y_2^{(o)} - y_1^{(o)}) \geq 0; \quad \forall\, q_2 \in [q_2^{(N)}, q_2^{(N)} + \varepsilon]. \tag{46}$$

The least favorable case is thus $q_1^{(o)} = q_1^{(N)}$ and $q_2^{(o)} = q_2^{(N)} + \varepsilon$ if $y_2^{(o)} \geq y_1^{(o)}$ at this point; it is $q_1^{(o)} = q_1^{(N)} + \varepsilon$ and $q_2^{(o)} = q_2^{(N)}$ if $y_2^{(o)} \leq y_1^{(o)}$ at this point; otherwise the solution must be a point on $\mathcal{B}_{\mathcal{X}_2}$ satisfying $y_1^{(o)} = y_2^{(o)}$.

APPENDIX

Proof of Lemma 1: Define

$$e_t = x_t - \hat{x}_t \tag{A1}$$

Then, combining (1), (2), (5), and (6) with (A1):

$$\frac{d}{dt}\begin{bmatrix} x_t \\ e_t \end{bmatrix} = \begin{bmatrix} A - BG & BG \\ 0 & A - HC \end{bmatrix}\begin{bmatrix} x_t \\ e_t \end{bmatrix} + \begin{bmatrix} \xi_t \\ \xi_t - H\theta_t \end{bmatrix} \tag{A2}$$

Since $(A - HC)$ is stable, the processes x_t and e_t are ergodic. Hence the cost can be written as

$$J(L,\Xi,\Theta) = \lim_{t \to \infty} E(x_t^T Qx_t + u_t^T Ru_t) \tag{A3}$$

Substituting (A1) and (5) in (A3) gives:

$$J = \lim_{t \to \infty} E\{x_t^T(Q + G^TRG)x_t - x_t^T G^TRGe_t - e_t^TG^TRGx_t + e_t^TG^TRGe_t\} \tag{A4}$$

Use of a simple trace identity converts (A4) into:

$$J = \lim_{t \to \infty} \mathrm{tr} \begin{bmatrix} Q + G^TRG & -G^TRG \\ -G^TRG & G^TRG \end{bmatrix} E\left\{\begin{bmatrix} x_t \\ e_t \end{bmatrix}\begin{bmatrix} x_t^T & e_t^T \end{bmatrix}\right\} \tag{A5}$$

Define

$$\bar{Q} = \begin{bmatrix} Q + G^TRG & -G^TRG \\ -G^TRG & G^TRG \end{bmatrix}$$

$$\Sigma = \lim_{t \to \infty} E\left\{\begin{bmatrix} x_t \\ e_t \end{bmatrix}\begin{bmatrix} x_t \\ e_t \end{bmatrix}^T\right\}$$

$$\bar{A} = \begin{bmatrix} A - BG & BG \\ 0 & A - HC \end{bmatrix}$$

$$\bar{X} = \begin{bmatrix} \Xi & \Xi \\ \Xi & \Xi + H\Theta H^T \end{bmatrix}$$

Then (A5) becomes

$$J = \mathrm{tr}\ \bar{Q}\ \Sigma \tag{A6}$$

where, in view of (A2), Σ is the unique solution of:

$$\bar{A}\ \Sigma + \Sigma\ \bar{A}^T + \bar{X} = 0 \tag{A7}$$

Since $\bar{A}$ is stable, Σ can also be written as:

$$\Sigma = \int_0^\infty e^{\bar{A}t}\ \bar{X}\ e^{\bar{A}^T t}\ dt \tag{A8}$$

Hence, (A6) becomes

$$J = \mathrm{tr}\ \bar{Q} \int_0^\infty e^{\bar{A}t}\ \bar{X}\ e^{\bar{A}t}\ dt \tag{A9}$$

With a few manipulations, (A9) can be rewritten as

$$J = \mathrm{tr}\ \bar{X} \int_0^\infty e^{\bar{A}^T t}\ \bar{Q}\ e^{\bar{A}^T t}\ dt \tag{A10}$$

Define

$$\bar{K} = \int_0^\infty e^{\bar{A}t^T}\ \bar{Q}\ e^{\bar{A}t}\ dt \tag{A11}$$

then

$$J = \mathrm{tr}\ \bar{X}\ \bar{K} \tag{A12}$$

where

$$\bar{A}^T\ \bar{K} + \bar{K}\ \bar{A} + \bar{Q} = 0 \tag{A13}$$

Let

$$\bar{K} = \begin{bmatrix} K_1 & K_2 \\ K_2^T & X \end{bmatrix}$$

Using the definitions of $\bar{A}$, $\bar{K}$ and $\bar{Q}$ leads to three equations:

$$(A - BG)^T K_1 + K_1 (A - BG) + G^T RG + Q = 0 \tag{A14}$$

$$(A - BG)^T K_2 + K_2 (A - HC) + K_1 BG - G^T RG = 0 \tag{A15}$$

$$(A - HC)^T X + X(A - HC) + G^T RG + K_2^T BG + G^T B^T K_2 = 0 \tag{A16}$$

Combining (A14) with (8) gives

$$K = K_1 \tag{A17}$$

Combining (A17), (A15) and (7) gives

$$K_2 = 0 \tag{A18}$$

Hence,

$$\bar{K} = \begin{bmatrix} K & 0 \\ 0 & X \end{bmatrix} \tag{A19}$$

where X (by virtue of (A16) and (A17) is given by (15). Finally, substituting (A19) into (A12) and using the definition of $\bar{X}$ yields (14).

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MINIMAX LINEAR SMOOTHING FOR CAPACITIES

[SMOOTHING FOR CAPACITIES]

by

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MINIMAX LINEAR SMOOTHING FOR CAPACITIES

Summary. Minimax linear smoothers are considered for the problem of estimating a homogeneous signal field in an additive orthogonal noise field. A minimax game with the quadratic-mean estimation error as an objective function is used to formulate this problem. Uncertainty in signal and noise field spectra is modeled using general nonparametric classes of measures proposed by Huber and Strassen for the problem of minimax hypothesis testing. These classes, which are described in terms of Choquet alternating capacities of order 2, include the conventional models for spectral uncertainty and admit a general solution to the minimax linear smoothing problem.

1. Introduction. Suppose we observe the random field $\{Y_z; z \in \mathbb{R}^n\}$ given for each $z \in \mathbb{R}^n$ by $Y_z = (S_z + N_z)$ where $\{S_z; z \in \mathbb{R}^n\}$ and $\{N_z; z \in \mathbb{R}^n\}$ are orthogonal random fields, each of which is second order, homogeneous, and quadratic-mean continuous. Suppose further that h is a complex-valued Borel-measurable function on $\mathbb{R}^n$, and that $\hat{S}_z$ denotes that the linear estimate of S_z based on $\{Y_z; z \in \mathbb{R}^n\}$ which has transfer function h. Then the quadratic-mean estimation error associated with $\hat{S}_z$ is given by

$$E\{|S_z - \hat{S}_z|^2\} = (2\pi)^{-n}[\int_{\mathbb{R}} |1-h|^2 dm_S + \int_{\mathbb{R}^n} |h|^2 dm_N] \triangleq e(h; m_S, m_N) \qquad (1)$$

where m_S and m_N are the spectral measures on $(\mathbb{R}^n, \mathcal{B}^n)$ associated (via Bochner's theorem [1, p. 245]) with $\{S_z; z \in \mathbb{R}^n\}$ and $\{N_z; z \in \mathbb{R}^n\}$, respectively. For fixed m_S and m_N, the minimum possible value of $e(h; m_S, m_N)$ is achieved by the estimate with transfer function $\hat{h} = dm_S/d(m_S + m_N)$ and this minimum value is given by $(2\pi)^{-n}\int_{\mathbb{R}} \hat{h}\, dm_N$.[1] If, on the other hand, m_S and m_N are known only to be in classes $\mathcal{M}_S$ and $\mathcal{M}_N$, respectively, of spectral measures on $(\mathbb{R}^n, \mathcal{B}^n)$, then a reasonable design strategy is to find a linear estimate whose transfer function minimizes $\sup_{\mathcal{M}_S \times \mathcal{M}_N} e(h; m_S, m_N)$. Such an estimate will be a minimax linear smoother for $\mathcal{M}_S$ and $\mathcal{M}_N$. Certain aspects of this problem have been considered by Kassam and Lim [2] and by the author [3]. In this paper we consider the minimax linear smoothing problem for the situation in which the measure classes $\mathcal{M}_S$ and $\mathcal{M}_N$ are of the type generated by 2-alternating capacities as considered by Huber and Strassen [4] in the context of minimax hypothesis testing. Examples of this type of class include mixtures, Prohorov and Kolmogorov (variational) neighborhoods, and other previously considered models for spectral uncertainty.

[1] Note that $e(h; m_S, m_N) = (2\pi)^{-n}\int_{\mathbb{R}^n} \hat{h}\, dm_N + (2\pi)^{-n}\int_{\mathbb{R}^n} |\hat{h} - h|^2 d(m_S + m_N)$.

Here we apply the results of Huber and Strassen to find the structure of minimax linear smoothers for general models of this type.

2. The minimax smoother for capacity classes. In the following, Ω denotes a fixed subset of $\mathbf{R}^n$, $\mathcal{A}$ denotes the Borel σ-algebra on Ω, and $\mathcal{M}$ denotes the class of all finite measures on $(\Omega,\mathcal{A})$. Recall that a finite set function v on $\mathcal{A}$ is a 2-alternating capacity (see Choquet [5]) on $(\Omega,\mathcal{A})$ if it is increasing, continuous from below, continuous from above for closed sets, and if it satisfies $v(\emptyset) = 0$ and $v(A \cup B) + (A \cap B) \leq v(A) + v(B)$ for all $A, B \in \mathcal{A}$. For a 2-alternating capacity v on $(\Omega,\mathcal{A})$ define the set $\mathcal{M}_v$ by

$$\mathcal{M}_v = \{m \in \mathcal{M} \mid m(A) \leq v(A) \text{ for all } A \in \mathcal{A}, \text{ and } m(\Omega) = v(\Omega)\}. \tag{2}$$

A number of properties of classes of the form of (2) have been developed by Huber and Strassen [4]. Note, for example, that $\mathcal{M}_v$ is weakly compact and that, if v is a measure, then $\mathcal{M}_v = \{v\}$.

For any pair (v_0,v_1) of 2-alternating capacities on $(\Omega,\mathcal{A})$ there exists a Radon-Nikodym derivative dv_1/dv_0, introduced in [4], which has the defining property that, for each $t \in [0,\infty]$,

$$r_t(\{dv_1/dv_0 > t\}) = \inf_{A \in \mathcal{A}} r_t(A) \tag{3}$$

where $r_t(A) \triangleq (1+t)^{-1}[tv_0(A) + v_1(A^c)]$. This derivative (which is a family of functions having the defining property (3)) is the basis for the minimax tests between capacity classes of the form of (2) as considered in [4]. Further properties and a generalization of this derivative have been considered by Rieder [6]. In this context we state the following result which is Theorem 4.1 of [4]:

Lemma 2.1 (Huber-Strassen): Suppose v_S and v_N are 2-alternating capacities and π_0 is a version dv_S/dv_N. Then there exist measures $q_S \in \mathcal{M}_{v_S}$ and $q_N \in \mathcal{M}_{v_N}$ such that $\pi_0 \in dq_S/dq_N$ and such that

$$q_S(\{\pi_0 < t\}) = v_S(\{\pi_0 < t\})$$

and

$$q_N(\{\pi_0 > t\}) = v_N(\{\pi_0 > t\})$$

for all $t \in [0,\infty]$.

Let $\mathcal{H}$ denote the class of all complex-valued $\mathcal{A}$-measurable functions on Ω. Lemma 2.1 leads to the following theorem:

Theorem 2.2: Suppose v_S and v_N are 2-alternating capacities on $(\Omega,\mathcal{A})$. Let π_0 be a version of dv_S/dv_N and choose (q_S,q_N) as in Lemma 2.1. Define $h_0 = \pi_0(1+\pi_0)^{-1}$. Then $[h_0,(q_S,q_N)]$ is a saddle-point solution to the game

$$\min_{h\in\mathcal{H}} \sup_{(m_S,m_N)\in\mathcal{M}_{v_S}\times\mathcal{M}_{v_N}} e(h;m_S,m_N)$$

where e is defined in (1), and thus h_0 is a minimax linear smoother for $\mathcal{M}_{v_S}$ and $\mathcal{M}_{v_N}$.

Proof: Noting that $h_0 \in dq_S/d(q_S+q_N)$, we have directly that

$$e(h_0;q_S,q_N) \le e(h;q_S,q_N)$$

for all $h \in \mathcal{H}$. Thus, it is sufficient to show

$$e(h_0;m_S,m_N) \le e(h_0;q_S,q_N) \tag{4}$$

for all $(m_S, m_N) \in \mathcal{M}_S \times \mathcal{M}_N$. Lemma 2.1 asserts that π_0 is stochastically smallest over $\mathcal{M}_{v_S}$ under q_S and is stochastically largest over $\mathcal{M}_{v_N}$ under q_N. Thus, since $|1-h_0|^2 = (1+\pi_0)^{-2}$ is decreasing in π_0 and $|h_0|^2 = \pi_0^2(1+\pi_0)^{-2}$ is increasing in π_0, we have

$$\int_\Omega |1-h_0|^2 dm_S \le \int_\Omega |1-h_0|^2 dq_S$$

and

$$\int_\Omega |h_0|^2 dm_N \le \int_\Omega |h_0|^2 dq_N$$

for all $(m_S, m_N) \in \mathcal{M}_{v_S} \times \mathcal{M}_{v_N}$. Equation (4) and hence Theorem 2.1 follow.

Note that, in view of Theorem 2.1, the pair (q_S, q_N) singled out by Lemma 2.1 can be thought of as a least-favorable pair of spectral measures for minimax linear smoothing. Concerning this pair of measures, we may also state the following property:

<u>Theorem 2.3</u>: The pair $(q_S, q_N) \in \mathcal{M}_{v_S} \times \mathcal{M}_{v_N}$ satisfies the conclusion of Lemma 2.1 if and only if its maximizes

$$\min_{h \in \mathcal{H}} e(h; m_S, m_N) = (2\pi)^{-n} \int_\Omega [dm_S/d(m_S+m_N)]\, dm_N$$

over all $(m_S, m_N) \in \mathcal{M}_{v_S} \times \mathcal{M}_{v_N}$.

<u>Proof</u>: Define $f = dm_N/d(m_S+m_N)$. Then

$$\min_{h \in \mathcal{H}} e(h; m_S, m_N) = (2\pi)^{-n} \int_\Omega f\, dm_S = (2\pi)^{-n} \int_\Omega (f - f^2)\, d(m_S + m_N).$$

Since $C[x] = (x - x^2)$ is concave and twice continuously differentiable on $[0,1]$, Theorem 2.3 follows from Theorem 6.1 of [illegible].

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3. <u>Discussion</u>. Theorem 2.2 gives the general solution to the minimax linear smoothing problem for signal and noise uncertainty classes of the form of (2). Several useful examples of classes of this type are given by Huber and Strassen in [4], and other useful examples are given by Rieder [6], Strassen [7], and Vastola and Poor [8]. Some of the most commonly used examples of classes of the form $\mathcal{M}_v$ can be written as ϵ-neighborhoods of some nominal measure μ. Examples of capacity classes that have this structure are contaminated mixtures, variational neighborhoods, and Prohorov neighborhoods (see [4]). For this type of class, an uncertainty model will consist of a nominal pair (μ_S, μ_N) of signal and noise spectral measures with respective degrees ϵ_S and ϵ_N of uncertainty placed on the nominal measures. The derivative between capacities generating classes of this type is often of the form (see Huber [9, 10] and Rieder [6])

$$\pi_0(\omega) = \max\{c', \min\{c'', \lambda(\omega)\}\}, \ \omega \in \Omega, \tag{5}$$

where λ is the Radon-Nikodym derivative between the nominal pair of measures (i.e., $\lambda \in d\mu_S/d\mu_N$) and c' and c'' are nonnegative constants with $c' \le c''$. If π_0 of (5) is a version of dv_S/dv_N, then Theorem 2.2 implies that a minimax linear smoother for $\mathcal{M}_{v_S}$ and $\mathcal{M}_{v_N}$ is given by

$$h_0(\omega) = \max\{k', \min\{k'', h'(\omega)\}\}, \ \omega \in \Omega \tag{6}$$

where $k' = c'/(1+c')$, $k'' = c''/(1+c'')$ and $h' = \lambda/(1+\lambda)$. Note that h' is the optimum smoother for the nominal model, and thus the minimax linear smoother for this case desensitizes the nominal smoother (to a degree depending on ϵ_S and ϵ_N) in those spectral regions where either μ_S or μ_N is dominant (i.e., where h' is near 1 or is near 0).

In the situations for which (5) is valid, (6) gives the transfer function of the minimax linear smoother. Suppose, for example, that $n = 1$, $\Omega = [-b,b]$ for some $b < \infty$, $c' < c''$, and h' is symmetric about $\omega = 0$ and is strictly decreasing on $[0,b]$. Then the minimax linear estimate of S_z determined by h_0 is given explicitly by

$$\hat{S}_z = \int_{-\infty}^{\infty} \tilde{h}_0(z - t) Y_t \, dt$$

where $\tilde{h}_0 \triangleq \mathcal{F}^{-1}\{h_0\}$ is given by

$$\tilde{h}_0(t) = \tilde{h}'(t) + k'[\sin(bt) - \sin(a't)]/(\pi t) + k''\sin(a''t)/(\pi t)$$
$$- \int_{-\infty}^{\infty} \tilde{h}'(t - \tau)[\sin(b\tau) - \sin(a'\tau) + \sin(a''\tau)](\pi\tau)^{-1} d\tau$$

with $\tilde{h}' = \mathcal{F}^{-1}\{h'\}$ and with a' [resp., a''] the positive solution to $h'(a') = k'$ [resp., $h'(a'') = k''$].

As a final comment we note that, although we assumed initially that the observation field was a continuous-parameter field, Theorems 2.2 and 2.3 are also directly applicable to the case in which the observation field is a discrete-parameter field (i.e., in which the time set is $\mathbb{Z}^n$) since this latter situations corresponds to the particular case of the analysis of Section 2 in which $\Omega = [-\pi,\pi]^n$.

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Minimax State Estimation for Linear Stochastic Systems with Noise Uncertainty

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Abstract—The problem of minimax linear state estimation for linear stochastic systems driven and observed in noises whose second-order properties are unknown is considered. Two general aspects of this problem are treated: the single-variable problem with uncertain noise spectra and the multivariable problem with uncertain componentwise noise correlation. General minimax results are presented for each of these situations involving characterizations of the minimax filters in terms of least favorable second-order properties. Explicit solutions are given for the spectral-band uncertainty model in the single-variable cases treated and for a matrix-norm neighborhood model in the multivariable case. Characterization of saddlepoints in terms of the extremal properties of the noise uncertainty classes is also discussed.

I. Introduction

Several recent studies have considered problems of minimax linear filtering and smoothing of stationary processes with uncertain second-order statistics. Examples are the works of Kassam and Lim [1], Kassam, Lim, and Cimini [2], and Poor [3], [4]. In this paper, we consider the related problem of minimax state estimation for linear stochastic systems in which the process noise and/or observation noise processes have uncertain second-order properties. In particular, we consider the usual linear system model

$$\dot{x}_t = Ax_t + B\xi_t; \qquad t \geq t_0 \tag{1}$$

$$y_t = Cx_t + \theta_t; \qquad t \geq t_0 \tag{2}$$

and $x_{t_0} = x_o$, where, for each $t \geq t_0$, $x_t \in \mathbb{R}^n$, $y_t \in \mathbb{R}^r$, $\xi_t \in \mathbb{R}^m$, and A, B, and C are constant matrices of the required dimensions. We assume throughout that $\{\xi_t;\ t \in \mathbb{R}\}$ and $\{\theta_t;\ t \in \mathbb{R}\}$ are orthogonal, zero-mean, wide-sense-stationary random processes.

In this paper we consider two general aspects of the problem of linear state estimation in (1) and (2) for situations in which the second-order statistics of the noise processes are specified only to be within some nonparametric classes. Specifically, in Section II we consider steady-state filtering for the single-variable case of (1) and (2). We assume that one of the noise processes $\{\xi_t;\ t \in \mathbb{R}\}$ and $\{\theta_t;\ t \in \mathbb{R}\}$ is a white noise and that the other has an unknown spectral density. A minimax mean-square error design criterion is adopted and the existence of a solution for this formulation is demonstrated for several useful noise spectral classes. In Section III we consider the multivariable case of (1) and (2) in which $\{\xi_t;\ t \in \mathbb{R}\}$ and $\{\theta_t;\ t \in \mathbb{R}\}$ are both multidimensional white noises with uncertain componentwise correlation matrices. A general minimax theorem which extends a result of Morris [5] is presented for this case and results are given which characterize a least-favorable correlation structure specifying the minimax filter.

II. One-Dimensional Filtering with Unknown Noise Spectra

A. General Formulation

In this section we consider the particular case of (1) and (2) in which $m = n = r = 1$ and $A < 0$. Without loss of generality we take $C = 1$. For a particular $t > t_0$, we consider the estimation of the state x_t based on the observation of $\{y_\tau;\ t_0 \leq \tau \leq t\}$ and consider the steady-state case resulting from the limit $t_0 \to -\infty$. We assume that the processes $\{\xi_t;\ t \in \mathbb{R}\}$ and $\{\theta_t;\ t \in \mathbb{R}\}$ have spectral densities σ_ξ and σ_θ, respectively. Note that the mean-square filtering error associated with a time-invariant linear filter whose transfer function is H is given by

$$E\{|x_t - \hat{x}_t|^2\} = (2\pi)^{-1}\int_{-\infty}^{\infty} |1 - H(\omega)|^2 B^2(\omega^2 + A^2)^{-1}\sigma_\xi(\omega)\,d\omega + (2\pi)^{-1}\int_{-\infty}^{\infty} |H(\omega)|^2 \sigma_\theta(\omega)\,d\omega \triangleq \mathcal{E}(H; \sigma_\xi, \sigma_\theta). \tag{3}$$

Let $\mathcal{H}^+$ denote the class of complex-valued transfer functions of causal time-invariant linear filters. Then for fixed σ_ξ and σ_θ, the optimum linear (minimum mean-square error) state estimation filter is found by solving the problem

$$\min_{H \in \mathcal{H}^+} \mathcal{E}(H; \sigma_\xi, \sigma_\theta). \tag{4}$$

If σ_ξ and σ_θ are such that the observation spectrum $\sigma_y(\omega) = [B^2(\omega^2 + A^2)^{-2}\sigma_\xi(\omega) + \sigma_\theta(\omega)]$ satisfies the Paley–Wiener condition,[1] then the solution to (4) is given by (Wong [6])

$$H^+(\omega) = \frac{1}{\sigma_y^+(\omega)}\left[\frac{B^2(\omega^2 + A^2)^{-1}\sigma_\xi(\omega)}{\sigma_y^-(\omega)}\right]_+ \tag{5}$$

where superscripts denote multiplicative spectral decomposition and subscripts denote additive spectral decomposition. If, on the other hand, the spectra σ_ξ and σ_θ are not known exactly, but are known to be in classes $\mathfrak{X}$ and $\mathfrak{N}$, respectively, of spectral densities, then an alternate design criterion to (4) is the minimax mean-square error criterion

$$\min_{H \in \mathcal{H}^+}\left\{\max_{(\sigma_\xi, \sigma_\theta) \in \mathfrak{X} \times \mathfrak{N}} \mathcal{E}(H; \sigma_\xi, \sigma_\theta)\right\}. \tag{6}$$

Several studies have considered problems related to (6). The analogous noncausal (smoothing) case has been considered for spectral band models in [1], for more general spectral density models in [3], and for general classes of spectral measures in [4]. Some aspects of a related causal case have been considered in [7] and more recently in [3]. Note that a saddlepoint solution to (6) is a point $(H'; \sigma_\xi', \sigma_\theta') \in \mathcal{H}^+ \times \mathfrak{X} \times \mathfrak{N}$ satisfying

$$\max_{(\sigma_\xi, \sigma_\theta) \in \mathfrak{X} \times \mathfrak{N}} \mathcal{E}(H'; \sigma_\xi, \sigma_\theta) = \mathcal{E}(H'; \sigma_\xi', \sigma_\theta') = \min_{H \in \mathcal{H}^+} \mathcal{E}(H; \sigma_\xi', \sigma_\theta'). \tag{7}$$

That is, a point satisfying (7) consists of a least-favorable spectral pair $(\sigma_\xi', \sigma_\theta') \in \mathfrak{X} \times \mathfrak{N}$ and their corresponding optimal (minimum-mean-square-error) filter H', which is the minimax-mean-square-error filter for $\mathfrak{X} \times \mathfrak{N}$. In the following subsections we consider the existence of such solutions for situations in which one of $\{\xi_t;\ t \in \mathbb{R}\}$ or $\{\theta_t;\ t \in \mathbb{R}\}$ represents white noise and the other has an uncertain spectral density.

B. White Process Noise with Uncertain Observation Noise

Suppose, for now, that the process noise $\{\xi_t;\ t \in \mathbb{R}\}$ represents white noise with known spectral height Ξ and that $\{\theta_t;\ t \in \mathbb{R}\}$ has a spectral density σ_θ which is known only to be in a class $\mathfrak{N} \subseteq L_1(\mathbb{R})$ of noise spectra where $L_1(\mathbb{R})$ denotes the class of absolutely integrable real-valued functions on $\mathbb{R}$. This problem now fits into the framework developed in [3]. In particular, we have the following result which is analogous to Theorem 2 of [3].

Theorem 1: Suppose $\mathfrak{N}$ is a convex class of spectral densities each satisfying the Paley–Wiener condition and $\mathfrak{X} = \{\sigma_\xi'\}$ where $\sigma_\xi'(\omega) = \Xi$ for all $\omega \in \mathbb{R}$. Then $(H'; \sigma_\xi', \sigma_\theta')$ with H' optimal for $(\sigma_\xi', \sigma_\theta')$ is a saddlepoint solution to (6) if, and only if, σ_θ' solves

$$\max_{\sigma_\theta \in N_\theta} J(\sigma_\theta) \tag{8}$$

where the functional J is defined by

$$J(\sigma) = \int_{-\infty}^{\infty} \sigma_x(\omega)\log[1 + \sigma(\omega)/\sigma_x(\omega)]\,d\omega \tag{9}$$

where σ_x is the state spectral density given by

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[1] Recall that the Paley–Wiener condition for σ is the finiteness of the integral $\int_{-\infty}^{\infty} |\log \sigma(\omega)|(\omega^2 + 1)^{-1}\,d\omega$.

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$$\sigma_x(\omega)=B^2\Xi(\omega^2+A^2)^{-1}. \tag{10}$$

Proof: It follows from the results of Yao [8] that, for any $\sigma_\theta\in\mathfrak{N}$, we have

$$\min_{H\in\mathcal{X}^+}\mathcal{E}(H;\sigma_\theta,\sigma_\xi')=(K/2)\left(1-\exp\left(-(\pi/K)J(\sigma_\theta)\right)\right) \tag{11}$$

where $K=B^2\Xi|A|$. Since the quantity in (11) is monotonically increasing in $J(\sigma_\theta)$ the "only if" part of Theorem 1 follows immediately. The "if" part follows as a straightfoward modification to the proof of Theorem 2 of [3].

Theorem 1 implies that a minimax state estimator for this case can be sought by considering the maximization problem of (8). With respect to this problem we have the following result from [4, Proposition 2]:

Theorem 2: The functional $J(\sigma)$ is upper-semicontinuous on $L_1(\mathbb{R})$ and thus achieves its maximum on any compact subset $\mathfrak{N}$.

Thus, if $\mathfrak{N}$ is compact, the existence of a solution to (6) is assured. Compactness is, of course, only sufficient (and not necessary) and may be a somewhat restrictive condition here since $L_1(\mathbb{R})$ has noncompact subsets which are of interest as spectral uncertainty models.

Example (Spectral Band Model): A spectral uncertainty model for which the robust *noncausal* filtering problem has been solved is the spectral band model. This is the model for spectral uncertainty used by Kassam and Lim [1], and for our case is given by

$$\mathfrak{N}=\left\{\sigma_\theta\,|\,\sigma_L(\omega)\le\sigma_\theta(\omega)\le\sigma_U(\omega);\ \omega\in\mathbb{R},\ (2\pi)^{-1}\int_{-\infty}^{\infty}\sigma_\theta(\omega)\,d\omega=P_\theta\right\} \tag{12}$$

where $P_\theta<\infty$ is a fixed noise power and σ_L and σ_U are fixed power spectra satisfying the Paley-Wiener condition. The solution to the problem of (8) can be found for this case by applying the results of [3] concerning the relationship of the minimax filtering problem to an analogous problem in hypothesis testing. It follows from the results of Section III of [3] and from the concavity of J that the spectrum σ_θ' maximizing J over $\mathfrak{N}$ of (12) is given by

$$\sigma_\theta'=2\pi P_\theta q' \tag{13}$$

where q' is a probability density in the class $\mathcal{P}_0=\{p\,|\,p=\sigma_\theta/(2\pi P_\theta);\ \sigma_\theta\in\mathfrak{N}\}$ that is least favorable (in the sense of Huber [9]) for testing $\mathcal{P}_0$ versus the probability density $p_1=\sigma_s/\int_{\mathbb{R}}\sigma_s$. A recent result of Kassam [10] gives q' for this case, and thus it follows straightforwardly from [10] and (13) that σ_θ' is given by

$$\sigma_\theta'(\omega)=\begin{cases}\sigma_L(\omega); & \text{if } \alpha\sigma_s(\omega)<\sigma_L(\omega)\\ \alpha\sigma_s(\omega); & \text{if } \sigma_L(\omega)\le\alpha\sigma_s(\omega)\le\sigma_U(\omega)\\ \sigma_U(\omega); & \text{if } \alpha\sigma_s(\omega)>\sigma_U(\omega)\end{cases} \tag{14}$$

where α is a constant chosen to satisfy $\int_{\mathbb{R}}\sigma_\theta'=2\pi P_\theta$. Note that all members of $\mathfrak{N}$ of (12) must satisfy the Paley-Wiener condition, and thus Theorem 1 implies that the minimax filter H' for this case is the minimum mean-square error filter for the pair $(\sigma_\theta',\sigma_\xi')$ given by (5) where $\sigma_\xi'(\omega)=\Xi$ for all $\omega\in\mathbb{R}$.

Note that, in general, σ_θ' of (14) will not yield a rational observation spectrum σ_y, so that the minimax transfer function H' must be found numerically. However, since H' represents the causal projection of the noncausal estimate of x_t (i.e., the estimate based on $\{y_\tau;\ \tau\in\mathbb{R}\}$), some insight about the structure of H' is derived by considering the corresponding noncausal estimate. The transfer function of the noncausal estimate for the spectral pair $(\sigma_\theta',\sigma_\xi')$ is $H_{nc}=\sigma_x/(\sigma_x+\sigma_\theta')$ where σ_x is from (10). Note that this transfer function represents a minimax smoother for this case (see [1]). Using (14) we note that H_{nc} is given by

$$H_{nc}(\omega)=\min\left\{H_L(\omega),\max\{k,H_U(\omega)\}\right\}$$

where $k=(1+\alpha)^{-1}$, $H_L=\sigma_x/(\sigma_x+\sigma_L)$, and $H_L=\sigma_x/(\sigma_x+\sigma_U)$. Thus, $H_L\ge H_{nc}\ge H_U$, and H_{nc} is less sensitive than either H_L or H_U in regions where H_L is large or H_U is small. Note that, if $H_U(\omega)\le k$ for all $\omega\in\mathbb{R}$, then H_{nc} is effectively a version of H_L with the gain limited to k in regions where $H_L(\omega)>k$.

The above example illustrates the structure of the minimax filter for the spectral-band uncertainty model which has been applied frequently as a model for such uncertainty. Other models can be treated similarly. For example, the class consisting of all spectral densities with a given power which differ in L_1 norm from a nominal spectrum can be treated using the results of [3]. Another widely used model is the mixture model which is a modification of (12) corresponding roughly to the case $\sigma_U\to\infty$ and $\sigma_L=(1-\epsilon)\sigma_0$ where σ_0 is a nominal spectral model with $\int\sigma_0=2\pi P_\theta$ and ϵ a degree of spectral uncertainty chosen by the designer. The solution to $\max_{\sigma\in\mathfrak{N}}J(\sigma)$ for this latter case is given by (14) with the third alternative never occurring.

Before considering the case of unknown process noise statistics, we note one further point. In particular, suppose the process $\{\theta_t;\ t\in\mathbb{R}\}$ is mean-square continuous but does not have a spectral density. It follows from Bochner's theorem (Wong [6]) that $\{\theta_t;\ t\in\mathbb{R}\}$ has a spectral measure m_θ and from the Lebesgue decomposition theorem (see Royden [11]) that we can always write $m_\theta=l_\theta+s_\theta$ where l_θ is absolutely continuous with respect to Lebesgue measure and s_θ is singular with respect to Lebesgue measure. Moreover, Snyders [12] has shown that the minimum filtering error $\min_{H\in\mathcal{X}^+}E\{|x_t-\hat{x}_t|^2\}$ depends only on l_θ. Thus, for any class of spectral measures, we might restrict our attention to the class consisting of their absolutely continuous parts and Theorem 1 may still apply.

C. White Observation Noise with Uncertain Process Noise

We now consider the alternate case in which the observation noise $\{\theta_t;\ t\in\mathbb{R}\}$ represents white noise with spectral height $\Theta>0$ and the process noise $\{\xi_t;\ t\in\mathbb{R}\}$ has spectral density σ_ξ which is known only to be within a class $\mathfrak{X}$. We assume that $\mathfrak{X}$ is such that the class $\mathcal{L}=\{\ell(\omega)=\sigma(\omega)(\omega^2+A^2)^{-1},\ \sigma\in\mathfrak{X}\}$ is a subset of $L_1(\mathbb{R})$. Within this context we have the following analog to Theorem 1.

Theorem 3: Suppose $\mathfrak{X}$ is convex and is such that the members of $\mathcal{L}$ satisfy the Paley-Wiener condition. Define $\mathfrak{N}=\{\sigma_\theta'\}$ where $\sigma_\theta'(\omega)=\Theta$ for all $\omega\in\mathbb{R}$. Then $(H';\sigma_\theta',\sigma_\xi')$ with H' optimal for $(\sigma_\theta',\sigma_\xi')$ is a saddlepoint solution to (6) if, and only if, σ_ξ' solves

$$\max_{\sigma_\xi\in\mathfrak{X}}I(\sigma_\xi) \tag{15}$$

where the functional $I(\sigma)$ is defined by

$$I(\sigma)=(2\pi)^{-1}\Theta\int_{-\infty}^{\infty}\log\left[1+\sigma(\omega)B^2/\left(\Theta(\omega^2+A^2)\right)\right]d\omega. \tag{16}$$

Proof: We note first that, for any $\sigma_\xi\in\mathfrak{X}$, we have (see Yao [8])

$$\min_{H\in\mathcal{X}^+}\mathcal{E}(H;\sigma_\theta',\sigma_\xi)=I(\sigma_\xi). \tag{17}$$

Thus, the "only if" part follows immediately. To show the "if" part, suppose $\sigma_\xi'\in\mathfrak{X}$ solves (15). Note that the inequality $\log(1+x)\le x$ and the assumption $\mathcal{L}\subseteq L_1(\mathbb{R})$ imply that $I(\sigma_\xi')<\infty$. For $\sigma_\xi\in\mathfrak{X}$, it follows straightforwardly from the concavity of $\log(1+x)$ on $(0,\infty)$ and the monotone convergence theorem (Royden [11]) that

$$\partial I((1-\epsilon)\sigma_\xi'+\epsilon\sigma_\xi)/\partial\epsilon|_{\epsilon=0}=(2\pi)^{-1}\Theta B^2$$
$$\cdot\int_{-\infty}^{\infty}\left[\Theta(\omega^2+A^2)+B^2\sigma_\xi'(\omega)\right]^{-1}\left[\sigma_\xi(\omega)-\sigma_\xi'(\omega)\right]d\omega. \tag{18}$$

Suppose H' is the minimum mean-square error filter from (5) for the pair $(\sigma_\theta',\sigma_\xi')$. We have (Yao [8])

$$|1-H'(\omega)|^2=\Theta\left[1+B^2\sigma_\xi'(\omega)/\left(\Theta(\omega^2+A^2)\right)\right]^{-1};\qquad \omega\in\Omega. \tag{19}$$

Thus, (3), (18), and (19) imply

$$\partial I((1-\epsilon)\sigma_\xi'+\epsilon\sigma_\xi)/\partial\epsilon|_{\epsilon=0}=\mathcal{E}(H';\sigma_\xi,\sigma_\theta')-\mathcal{E}(H';\sigma_\xi',\sigma_\theta'). \tag{20}$$

Since $I(\sigma)$ is concave and σ_ξ' maximizes $I(\sigma)$ over $\mathfrak{X}$, we must have $\partial I((1-\epsilon)\sigma_\xi'+\epsilon\sigma_\xi)/\partial\epsilon|_{\epsilon=0}\le0$ for all $\sigma_\xi\in\mathfrak{X}$. Thus, (20) gives the left-hand equation of (7), and $(H';\sigma_\xi',\sigma_\theta')$ is a saddle-point solution to (6). This completes the proof. □

Theorem 3 implies that the solution to (6) may be sought for this case by seeking a solution to $\max_{\sigma_\xi\in\mathfrak{X}}I(\sigma_\xi)$. We note that, I is concave and can be shown to be upper-semicontinuous. Thus, the solution to (6) is assured if $\mathfrak{X}$ is compact. Again, if $\{\xi_t;\ t\in\mathbb{R}\}$ can have a spectral measure which is not absolutely continuous with respect to Lebesgue measure, the white-observation-noise case might still be treated via Theorem 3 by

restricting attention to the absolutely continuous parts of the relevant spectra.

As an example, it is again of interest to consider the spectral band model discussed above. However, it is more reasonable to assume that the power in the state process $\{x_t;\ t\in\mathbf{R}\}$ is known rather than the power in the process noise $\{\xi_t;\ t\in\mathbf{R}\}$. Thus, we consider the following modified version of (12):

$$\mathcal{X}=\Big\{\sigma_\xi \mid \sigma_L(\omega)\le\sigma_\xi(\omega)\le\sigma_U(\omega);\ \omega\in\Omega,$$

$$(2\pi)^{-1}B^2\int_{-\infty}^{\infty}\sigma_\xi(\omega)(\omega^2+A^2)^{-1}\,d\omega=P_x\Big\} \quad (21)$$

where $\sigma_L(\omega)(\omega^2+A^2)^{-1}$ and $\sigma_U(\omega)(\omega^2+A^2)^{-1}$ satisfy the Paley–Wiener condition. Then it is straightforward to show that the solution to max $I(\sigma)$ is given for this case by

$$\sigma_\xi'(\omega)=\begin{cases}\sigma_L(\omega); & \text{if } \beta(\omega^2+A^2)<\sigma_L(\omega)\\ \beta(\omega^2+A^2); & \text{if } \sigma_L(\omega)\le\beta(\omega^2+A^2)\le\sigma_U(\omega)\\ \sigma_U(\omega); & \text{if } \beta(\omega^2+A^2)>\sigma_U(\omega)\end{cases} \quad (22)$$

where β is chosen so that $(2\pi)^{-1}B^2\int_{-\infty}^{\infty}\sigma_\xi(\omega)(\omega^2+A^2)^{-1}d\omega=P_x$. The minimax filter is then given by (5) for the pair $(\sigma_\xi',\sigma_\theta')$.

Another situation of interest here is the case in which the process noise is white with unknown spectral height. That is, we have

$$\mathcal{X}=\{\sigma_\xi \mid \sigma_\xi(\omega)=\Xi \quad \text{for all} \quad \omega\in\Omega \quad \text{and some} \quad \Xi\in[a,b]\} \quad (23)$$

where $0\le a\le b<\infty$. It follows immediately from (16) that, for this case, $\sigma_\xi'(\omega)=b$, a solution which is more or less obvious even without Theorem 3. Note that a similar result would be obtained if we allowed $\mathcal{R}$ of Section II-B to be of the form of (23). In fact, we note that if either $\mathcal{R}$ of Section II-B or $\mathcal{X}$ contains an maximal element σ_E (i.e., an element $\sigma_E\in\mathcal{R}$ (or $\mathcal{X}$) such that $\sigma_E(\omega)\ge\sigma(\omega)$ for all $\omega\in\Omega$ and for all $\sigma\in\mathcal{R}$ (or $\mathcal{X}$)), then the corresponding least-favorable spectrum σ_θ' or σ_ξ' will be the maximal element. This problem is related to the notion of a bounding filter which is an alternate approach to treating problems of unknown noise statistics (see Nahi and Weiss [13], [14] and Greenlee and Leondes [15]). Unfortunately, maximals do not exist in many models for spectral uncertainty (for example, the spectral band models (12) and (21) do not generally contain maximals). However, for white-noise models the role of extremal points is more important as will be demonstrated below in the treatment of the multivariable filtering problem with white noise of uncertain componentwise correlation in both the process and observation.

III. Minimax State Estimation for Multivariable Systems

A. Formulation

The purpose of this section is to consider the state estimation problem for the multivariable form of the linear stochastic system (1) when the second-order statistics of the process and observation noises are not known. Let the dimensions of x_t, ξ_t, and y_t (n, m, and r, respectively) be arbitrary and assume that $\{\xi_t;\ t\in\mathbf{R}\}$ and $\{\theta_t;\ t\in\mathbf{R}\}$ are orthogonal, zero-mean, wide-sense-stationary white noise processes with covariances $E\{\xi_t\xi_\tau^T\}=\Xi\delta(t-\tau)$ and $E\{\theta_t\theta_\tau^T\}=\Theta\delta(t-\tau)$ where Ξ and Θ are symmetric positive definite matrices. Also, it will be assumed throughout this section that (A,C) is an observable pair and (A,B) is a controllable pair. This section will derive the minimax state estimator when the matrices Ξ and Θ are known only to be contained in subsets of the convex cones of positive definite matrices $S^{m\times m}\subset\mathbf{R}^{m\times m}$ and $S^{r\times r}\subset\mathbf{R}^{r\times r}$, respectively.

We will restrict ourselves to consideration of causal, linear time-invariant filters which produce a wide-sense-stationary error process

$$\epsilon_t\triangleq x_t-\hat{x}_t \quad (24)$$

where $\{\hat{x}_t;\ t\in\mathbf{R}\}$ is the output of the filter. Define $\mathcal{H}^+$ to be the space of complex valued $n\times r$ matrices of Laplace transforms of all such filters. For every $H\in\mathcal{H}^+$ and for every Ξ and Θ the weighted mean-square estimation error is defined as

$$\mathcal{E}(H;\Xi,\Theta)\triangleq E\{\epsilon_t^TQ\epsilon_t\} \quad (25)$$

where Q is a positive semidefinite matrix.

The Laplace transform of (1), the assumption $H\in\mathcal{H}^+$ and the assumption that $\{\xi_t;\ t\in\mathbf{R}\}$ and $\{\theta_t;\ t\in\mathbf{R}\}$ are orthogonal allows the weighted mean-square error (25) to be written as

$$\mathcal{E}(H;\Xi,\Theta)=(2\pi j)^{-1}\operatorname{tr}\Big\{Q\int_{-j\infty}^{j\infty}[I-H(s)C](sI-A)^{-1}$$

$$\cdot B\Xi B^T(-sI-A)^{-T}[I-H(-s)C]^T\,ds$$

$$+Q\int_{-j\infty}^{j\infty}H(s)\Theta H(-s)^T\,ds\Big\} \quad (26)$$

where the superscript T denotes the transpose of the indicated matrix, I denotes the $n\times n$ identity matrix, and tr denotes the trace of the bracketed matrix.

When Ξ and Θ are fixed the filter which minimizes (26) over $\mathcal{H}^+$ is given by the steady state Kalman–Bucy filter [16] (independently of the weighting matrix Q):

$$\dot{\hat{x}}_t=A\hat{x}_t+K[y_t-C\hat{x}_t] \quad (27)$$

$$K=PC^T\Theta^{-1} \quad (28)$$

$$0=AP+PA^T+\Xi-PC^T\Theta^{-1}CP. \quad (29)$$

The transfer function of the filter defined by (27)–(29) is given by

$$\overline{H}(s)=[sI-(A-KC)]^{-1}K. \quad (30)$$

The corresponding minimum weighted mean-square error is

$$\mathcal{E}(\overline{H};\Xi,\Theta)=\operatorname{tr}\{QP(\Xi,\Theta)\} \quad (31)$$

where the dependence of the error covariance matrix on Ξ and Θ is denoted explicitly.

Often the second-order statistics of $\{\xi_t;\ t\in\mathbf{R}\}$ and $\{\theta_t;\ t\in\mathbf{R}\}$ are not known exactly. A common representation of this type of uncertainty is that Ξ and Θ are contained in sets $\mathcal{X}\subset S^{m\times m}$ and $\mathcal{R}\subset S^{r\times r}$, respectively. The problem to be considered is to find the best filter in $\mathcal{H}^+$ in the sense that this filter minimizes the largest weighted mean-square error produced when Ξ and Θ range over all possible values. That is, we wish to solve the minimax problem

$$\min_{H\in\mathcal{H}^+}\ \max_{\substack{\Xi\in\mathcal{X}\\ \Theta\in\mathcal{R}}}\mathcal{E}(H;\Xi,\Theta). \quad (32)$$

B. Existence and Characterization of a Saddlepoint

Two important results concerning solutions to the minimax problem formulated in Section III-A are presented in this section. The first result establishes an equivalence between a saddlepoint solution to (32) and the Kalman–Bucy filter corresponding to a particular (Ξ,Θ) pair. The second result establishes the existence of a saddlepoint when the sets $\mathcal{X}$ and $\mathcal{R}$ are convex and compact.

To obtain these results, we will need the following well-known theorem (cf. [17]) which establishes the fact that the existence of a saddlepoint is a necessary and sufficient condition for the minimax problem (32) to be equivalent to the corresponding maximin problem

$$\max_{\substack{\Xi\in\mathcal{X}\\ \Theta\in\mathcal{R}}}\ \min_{H\in\mathcal{H}^+}\mathcal{E}(H;\Xi,\Theta). \quad (33)$$

Theorem 4: There exists a triplet $(H_0,\Xi_0,\Theta_0)\in\mathcal{H}^+\times\mathcal{X}\times\mathcal{R}$ satisfying the saddlepoint condition

$$\mathcal{E}(H_0;\Xi,\Theta)\le\mathcal{E}(H_0;\Xi_0,\Theta_0)\le\mathcal{E}(H;\Xi_0,\Theta_0)$$

$$\forall H\in\mathcal{H}^+,\ \Xi\in\mathcal{X},\ \Theta\in\mathcal{R} \quad (34)$$

if and only if the values of (32) and (33) are equal. Moreover, a triplet satisfying (34) is a solution to (32) and (33).

Theorem 5 provides the desired characterization of a saddlepoint.

Theorem 5: Assume that there exists $\Xi_0 \in \mathcal{X}$ and $\Theta_0 \in \mathcal{R}$ which satisfy

$$\operatorname{tr}\{B^T Z_0 B \Xi\} \le \operatorname{tr}\{B^T Z_0 B \Xi_0\} \qquad \forall \Xi \in \mathcal{X} \tag{35}$$

$$\operatorname{tr}\{K_0^T Z_0 K_0 \Theta\} \le \operatorname{tr}\{K_0^T Z_0 K_0 \Theta_0\} \qquad \forall \Theta \in \mathcal{R} \tag{36}$$

where K_0 is the Kalman–Bucy gain corresponding to Ξ_0 and Θ_0 defined by (28)–(29) and Z_0 is the solution to the equation

$$(A-K_0C)^T Z_0 + Z_0(A-K_0C) + Q = 0. \tag{37}$$

Let $H_0(s)$ be the transfer function of the Kalman–Bucy filter (30) corresponding to Ξ_0 and Θ_0. Then (H_0, Ξ_0, Θ_0) is a solution to (32) and (33).

Conversely, suppose that (H_0, Ξ_0, Θ_0) is a saddlepoint for (32). Then *H_0 is the transfer function of the Kalman–Bucy filter (30), and Ξ_0 and Θ_0 satisfy (35)–(37).*

Proof:

Sufficiency: Consider the maximin problem (33). For fixed Ξ and Θ, the solution of the minimization is given by the Kalman–Bucy filter (27)–(29). The filter transfer function $\bar{H}(s)$ is given by (30) and the mean-square error is given by [see (26)]

$$\mathcal{E}(\bar{H}; \Xi, \Theta) = (2\pi j)^{-1} \operatorname{tr}\left\{ Q \int_{-j\infty}^{j\infty} [I - \bar{H}(s)C](sI-A)^{-1} \cdot B\Xi B^T(-sI-A)^{-T}[I-\bar{H}(-s)C]^T ds + Q\int_{-j\infty}^{j\infty} \bar{H}(s)\Theta\bar{H}(-s)^T ds \right\}. \tag{38}$$

Since (38) is additively separable in Ξ and Θ, the joint maximization required by (33) can be carried out separately.

First, consider the process noise. Assume Ξ_0 satisfies (35) and (37) for fixed Θ. The solution to (37) is given by

$$Z_0 = \int_0^\infty e^{(A-K_0C)^T t} Q e^{(A-K_0C)t} dt. \tag{39}$$

Substituting (39) in (35) and using a simple trace identity gives

$$\operatorname{tr}\{B^T Z_0 B \Xi\} = \operatorname{tr}\left\{ Q \int_0^\infty e^{(A-K_0C)t} B \Xi B^T e^{(A-K_0C)^T t} dt \right\}. \tag{40}$$

By Parseval's theorem, (40) can be written as

$$\operatorname{tr}\{B^T Z_0 B \Xi\} = (2\pi j)^{-1} \operatorname{tr}\left\{ Q \int_{-j\infty}^{j\infty} [sI-(A-K_0C)]^{-1} B\Xi B^T[-sI-(A-K_0C)]^{-T} ds \right\}. \tag{41}$$

However,

$$\begin{aligned} [sI-(A-K_0C)]^{-1} &= [I-(sI-A+K_0C)^{-1}K_0C](sI-A)^{-1} \\ &= [I-\bar{H}(s)C](sI-A)^{-1}. \end{aligned} \tag{42}$$

Substituting (42) into (41) gives the first term on the right-hand side of (38). Then (35) implies (38) is maximized over $\mathcal{X}$, which establishes

$$\mathcal{E}(H_0; \Xi, \Theta) \le \mathcal{E}(H_0; \Xi_0, \Theta). \tag{43}$$

Now, assume Ξ is fixed and Θ_0 satisfies (36), (37). Substituting (39) into (36) and manipulating the trace gives

$$\operatorname{tr}\{K_0^T Z_0 K_0 \Theta\} = \operatorname{tr}\left\{ Q \int_0^\infty e^{(A-K_0C)t} K_0 \Theta K_0^T e^{(A-K_0C)^T t} dt \right\} \tag{44}$$

Again using Parseval's theorem, we obtain

$$\begin{aligned} \operatorname{tr}\{K_0^T Z_0 K_0 \Theta\} &= (2\pi j)^{-1} \operatorname{tr}\left\{ Q \int_{-j\infty}^{j\infty} [sI-(A-K_0C)]^{-1} \cdot K_0\Theta K_0^T[-sI-(A-K_0C)]^{-T} ds \right\} \\ &= (2\pi j)^{-1} \operatorname{tr}\left\{ Q \int_{-j\infty}^{j\infty} \bar{H}(s)\Theta\bar{H}(-s)^T ds \right\}. \end{aligned} \tag{45}$$

Thus, (36) implies (38) is minimized over $\mathcal{R}$ which establishes

$$\mathcal{E}(H_0; \Xi, \Theta) \le \mathcal{E}(H_0; \Xi, \Theta_0). \tag{46}$$

Inequalities (43) and (46) together with the aforementioned separability of (38) imply

$$\mathcal{E}(H_0; \Xi, \Theta) \le \mathcal{E}(H_0; \Xi_0, \Theta_0). \tag{47}$$

The upper inequality

$$\mathcal{E}(H_0; \Xi_0, \Theta_0) \le \mathcal{E}(H; \Xi_0, \Theta_0) \tag{48}$$

follows trivially from the fact that H_0 is the minimum mean-square error estimator. Thus, (H_0, Ξ_0, Θ_0) is a saddlepoint for (32) and (33) and, by Theorem 4, also a solution for (32) and (33).

Necessity: Suppose (H_0, Ξ_0, Θ_0) satisfies (34). By Theorem 4, (H_0, Ξ_0, Θ_0) solves (33). Hence H_0 is the transfer function of the Kalman–Bucy filter. Let $P(\Xi, \Theta) = E\{\epsilon_t \epsilon_t^T\}$ when (Ξ, Θ) are the second-order statistics. Then $P(\Xi, \Theta)$ is given by

$$0 = (A-K_0C)P(\Xi,\Theta) + P(\Xi,\Theta)(A-K_0C)^T + B\Xi B^T + K_0\Theta K_0^T. \tag{49}$$

Thus, the difference

$$\Delta P \triangleq P(\Xi,\Theta) - P(\Xi_0,\Theta_0) \tag{50}$$

is given by

$$0 = (A-K_0C)\Delta P + \Delta P(A-K_0C)^T + B(\Xi-\Xi_0)B^T + K_0(\Theta-\Theta_0)K_0^T. \tag{51}$$

Thus, ΔP is given by

$$\Delta P = \int_0^\infty e^{(A-K_0C)t}\left[B(\Xi-\Xi_0)B^T + K_0(\Theta-\Theta_0)K_0^T\right] e^{(A-K_0C)^T t} dt. \tag{52}$$

By (34)

$$\operatorname{tr}\{Q\Delta P\} = \mathcal{E}(H_0; \Xi, \Theta) - \mathcal{E}(H_0; \Xi_0, \Theta_0) \le 0. \tag{53}$$

After substituting (52) into (53), using a simple trace identity, and using the definition of Z_0 [(39) and (37)], we obtain

$$\operatorname{tr}\{Z_0(\Xi-\Xi_0)\} + \operatorname{tr}\{Z_0(\Theta-\Theta_0)\} \le 0 \qquad \forall \Xi \in \mathcal{X},\ \Theta \in \mathcal{R}. \tag{54}$$

In particular, $\Xi = \Xi_0$ implies (36) and $\Theta = \Theta_0$ implies (35). □

Thus, we see that conditions (35)–(37) are equivalent to the existence of a saddlepoint. If such a saddlepoint exists then the minimax filter is simply the Kalman–Bucy filter for the (Ξ_0, Θ_0) pair which satisfies (35)–(37). This result can be used to establish the existence of a saddlepoint.

Theorem 6: If $\mathcal{X}$ and $\mathcal{R}$ are convex, compact subsets of $S^{m\times m}$ and $S^{r\times r}$, respectively, then a saddlepoint solution for the minimax problem (32) exists.

Proof: The proof shows that a solution to the maximin problem (33) exists and satisfies conditions (35)–(37) of Theorem 5. By (26)–(31),

$$\min_{H \in \mathcal{X}^*} \mathcal{E}(H; \Xi, \Theta) = \operatorname{tr}\{QP(\Xi,\Theta)\} \tag{55}$$

is continuous in Ξ and Θ. Since $\mathcal{X}$ and $\mathcal{R}$ are compact, a solution to (33) exists. Let $(\bar{H}, \Xi_0, \Theta_0)$ be such a solution.

Since $\mathcal{S}(H;\Xi,\Theta)$ is concave in Ξ and Θ for every $H\in\mathcal{X}^+$ [see (26)], $\min_{H\in\mathcal{X}^+}\mathcal{S}(H;\Xi,\Theta)$ is also concave in Ξ and Θ. Hence, the Fréchet differential of (55) must be nonpositive in every direction into the set $\mathfrak{X}\times\mathfrak{N}$. Let $\delta P(\Xi_0,\Theta_0;\Delta\Xi,\Delta\Theta)$ denote the Fréchet differential of P evaluated at (Ξ_0,Θ_0) in the direction $(\Delta\Xi,\Delta\Theta)$. By (29),

$$0=(A-K_0C)\delta P+\delta P(A-K_0C)^T+B\Delta\Xi B^T+K_0\Delta\Theta K_0^T \tag{56}$$

where the dependence of δP on Ξ, Θ, $\Delta\Xi$, and $\Delta\Theta$ has been suppressed. Thus, δP is given by

$$\delta P=\int_0^\infty e^{(A-K_0C)t}\left[B\Delta\Xi B^T+K_0\Delta\Theta K_0^T\right]e^{(A-K_0C)^Tt}\,dt. \tag{57}$$

Using (57) and a simple trace identity, the Fréchet differential of (55) becomes

$$\operatorname{tr}\{Q\delta P(\Xi_0,\Theta_0;\Delta\Xi,\Delta\Theta)\}=\operatorname{tr}\{B^TZ_0B\Delta\Xi\}+\operatorname{tr}\{K_0^TZ_0K_0\Delta\Theta\} \tag{58}$$

where Z_0 satisfies (37).

Consider an arbitrary point $(\Xi,\Theta)\in\mathfrak{X}\times\mathfrak{N}$. Since $\mathfrak{X}$ and $\mathfrak{N}$ are convex, the line segment joining (Ξ_0,Θ_0) and (Ξ,Θ) is in $\mathfrak{X}\times\mathfrak{N}$ and hence,

$$(\Delta\Xi,\Delta\Theta)=(\Xi-\Xi_0,\Theta-\Theta_0) \tag{59}$$

is a direction into $\mathfrak{X}\times\mathfrak{N}$. Substituting (59) into (58) and requiring (58) to be nonpositive gives

$$\operatorname{tr}\{B^TZ_0B(\Xi-\Xi_0)\}+\operatorname{tr}\{K_0^TZ_0K_0(\Theta-\Theta_0)\}\leq 0. \tag{60}$$

The choice $(\Xi,\Theta)=(\Xi,\Theta_0)$ in (60) gives (35) while the choice $(\Xi,\Theta)=(\Xi_0,\Theta)$ in (60) gives (36). Thus, by Theorem 5, (Ξ_0,Θ_0) is a saddlepoint for (32). □

C. Discussion

Theorem 5 provides an equivalent characterization of a saddlepoint in terms of quantities associated with the Kalman filter of a particular (Ξ_0,Θ_0) pair. Theorem 6 establishes the existence of a saddlepoint when $\mathfrak{X}$ and $\mathfrak{N}$ are convex and compact. Taken together, Theorems 5 and 6 show that the minimax filter which solves (33) when $\mathfrak{X}$ and $\mathfrak{N}$ are convex and compact is given by the Kalman filter corresponding to the (Ξ_0,Θ_0) pair which maximizes $\operatorname{tr}\{QP(\Xi,\Theta)\}$.

There are two important cases for which the conditions (35)–(37) provide explicit solutions. The first occurs when the sets $\mathfrak{X}$ and $\mathfrak{N}$ have maximal elements,[2] and the second is when the sets $\mathfrak{X}$ and $\mathfrak{N}$ are each line segments.

Theorem 7: Suppose $\Xi_0\in\mathfrak{X}$ and $\Theta_0\in\mathfrak{N}$ are maximal elements in their respective sets. Then Ξ_0 and Θ_0 satisfy (35)–(37).

Proof: By the controllability and observability assumptions on the system (1) and (2), the closed-loop filter matrix $(A-K_0C)$ is asymptotically stable. Hence, Z_0 defined by (37) is positive definite. The maximality of Ξ_0 and Θ_0 thus implies (35)–(36). □

Theorems 5 and 7 combined imply that if the sets $\mathfrak{X}$ and $\mathfrak{N}$ each contain a maximal element then the Kalman–Bucy filter corresponding to those elements is a minimax filter. This result is a precise expression of the intuitive design procedure of using the worst-case noises. It generalizes the concept of a bounding filter (see Nahi and Weiss [14]) to the multivariable estimation problem. One application given by the following example demonstrates the intuitive nature of the result.

Example: A common method of modeling uncertainty in the second order *a priori* statistics of system (1) and (2) is to choose a nominal pair (Ξ_N,Θ_N) and assume that the true Ξ and Θ differ from the nominal in norm by no more than some positive constant η. Assume that the norm to be used for each matrix is the norm induced by the Euclidean vector norm on the underlying space (R^m and R^p, respectively). Define

$$\mathfrak{X}\triangleq\{\Xi:\|\Xi-\Xi_N\|\leq\eta,\ \Xi>0\} \tag{61}$$

$$\mathfrak{N}\triangleq\{\Theta:\|\Theta-\Theta_N\|\leq\eta,\ \Theta>0\}. \tag{62}$$

Then each set has a maximal element

$$\Xi_0=\Xi_N+\eta I \tag{63}$$

$$\Theta_0=\Theta_N+\eta I \tag{64}$$

and the minimax filter is the Kalman–Bucy filter corresponding to (Ξ_0,Θ_0). □

Another reasonable model for uncertainty in the second-order *a priori* statistics of system (1) and (2) is that the spectral density matrices are a convex combination of two possible nominals. Let

$$\mathfrak{X}\triangleq\{\Xi:\Xi=\lambda\Xi_1+(1-\lambda)\Xi_2;\ \Xi_1>0,\ \Xi_2>0,\ \lambda\in[0,1]\} \tag{65}$$

$$\mathfrak{N}\triangleq\{\Theta:\Theta=\gamma\Theta_1+(1-\gamma)\Theta_2;\ \Theta_1>0,\ \Theta_2>0,\ \gamma\in[0,1]\}. \tag{66}$$

Suppose that (Ξ_0,Θ_0) satisfy (35)–(37) and let λ_0 and γ_0 be the corresponding constants defined by (65) and (66). Then (35) and (36) become

$$(\lambda-\lambda_0)\operatorname{tr}\{B^TZ_0B(\Xi_1-\Xi_2)\}\leq 0 \tag{67}$$

$$(\gamma-\gamma_0)\operatorname{tr}\{K_0^TZ_0K_0(\Theta_1-\Theta_2)\}\leq 0. \tag{68}$$

Conditions (67) and (68) imply that (35) and (36) can be satisfied with strict inequality for all $\Xi\neq\Xi_0$ and $\Theta\neq\Theta_0$ only if Ξ_0 and Θ_0 are endpoints of the set. Otherwise, we must have

$$\operatorname{tr}\{B^TZ_0B(\Xi_1-\Xi_2)\}=0 \tag{69}$$

$$\operatorname{tr}\{K_0^TZ_0K_0(\Theta_1-\Theta_2)\}=0 \tag{70}$$

where Z_0 and K_0 depend on λ_0.

IV. Summary

Minimax approaches to two problems associated with the estimation of the state of a linear stochastic system have been considered. The scalar problem for which the noise spectrum is uncertain was examined in Section II. The minimax filter for this problem was characterized in terms of the least favorable noise spectrum. Section III considered a similar problem for the multivariable estimation problem when the noises are white processes with unknown *a priori* second-order statistics. The saddlepoint solution for this problem is given by a Kalman–Bucy filter for a least-favorable pair of componentwise correlation matrices.

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AN ANALYSIS OF THE EFFECTS OF SPECTRAL UNCERTAINTY ON WIENER FILTERING

[Spectral Uncertainty in Wiener Filtering]

by

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Abstract

Results from an extensive study of the performance of Wiener filtering under spectral uncertainty are presented. For a variety of spectral uncertainty models the Wiener filter is shown to have unacceptable sensitivity to even small deviations from those signal and noise spectral densities which were used to design the filter. Performance of a robust filter (designed to have the best possible performance when the uncertainty is worst) is also examined. In most cases the robust filter's insensitivity to spectral uncertainty makes it preferable to the traditional Wiener filter.

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1. Introduction

The solution to the traditional stationary linear (i.e., Wiener) filtering problem requires exact knowledge of the signal and noise spectra. Often in practice it is unrealistic to assume such knowledge. Despite this, Wiener filters are widely used for steady-state filtering. In this paper we consider the performance of Wiener filtering when the signal and noise spectra differ to a small degree from those assumed in the design process.

In particular, in Section 2 we consider the Wiener filter for a particular signal and noise spectral pair which would be natural to assume is the true spectral pair. We then look again at our circumstances and model the uncertainty we might have about our choice of spectra. In so doing we find that the potential exists for totally unacceptable performance degradation in the presence of even small degrees of uncertainty.

In Section 3 we consider filters termed "robust". These filters are designed to have the best "worst-case" performance over uncertainty classes of spectra. The method of design is due to Poor (1980) and was based on the work of Kassam and Lim (1977). As we will see, the advantage of these robust filters is that they are least sensitive in the sense that they have the smallest possible maximum deviation from optimality within the constraints imposed by our uncertainty.

Of course there is a trade-off involved in robust filtering. While the robust filter has better worst-case performance, we cannot expect it to have optimal performance should our original choice of spectra be the true ones. In Section 3 we will consider this trade-off as well.

2. *The Sensitivity of the Wiener Filter to Spectral Uncertainty*

The mean-square-error (MSE) for linear filtering of a signal in uncorrelated additive noise, where both signal and noise are modeled as real, zero-mean, second-order, wide-sense stationary random processes, is given by

$$e(\sigma,\nu;H) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\sigma(\omega)\,|1-H(\omega)|^2 + \nu(\omega)\,|H(\omega)|^2]\;d\omega\;, \tag{1}$$

where H is the transfer function of the filter and σ and ν are the power spectral densities (PSD's) of the signal and noise, respectively. For a fixed signal and noise spectral pair, (σ,ν), $e(\sigma,\nu;H)$ is minimized by the Wiener filter

$$H^*(\omega) = \frac{\sigma(\omega)}{\sigma(\omega) + \nu(\omega)} \tag{2}$$

and the minimum MSE is

$$e^*(\sigma,\nu) \triangleq e(\sigma,\nu;H^*) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H^*(\omega)\;\nu(\omega)\;d\omega\;. \tag{3}$$

Unfortunately, as we discussed in the introduction, it is often the case in practice that our knowledge of the signal and/or noise PSD's is inexact. If the σ and ν we choose for designing H^* are not the true spectra, then our filter will generally have less than optimal performance. To illustrate the degree of performance degradation that can result from such mis-modeling, we consider the following examples. The numerical results presented here and in the following section comprise a representative selection from an extensive numerical study (Vastola, 1981).

The p-point class. For a number of applications it is natural to

assume that we have a narrow-band first-order Markov signal in wide-band first-order Markov noise, i.e. that

$$\sigma_o(\omega) = \frac{2\alpha_S v_S^2}{\alpha_S^2 + \omega^2}$$

and (4)

$$\nu_o(\omega) = \frac{2\alpha_N v_N^2}{\alpha_N^2 + \omega^2}$$

where $\alpha_S \ll \alpha_N$ are the 3 dB bandwidths and v_S^2 and v_N^2 are the powers of the signal and noise, respectively. For Fig. 1 we have $\alpha_N = 10$ and $\alpha_S = 1$.

In the figures of this paper we have used a measure of performance which we refer to simply as output signal-to-noise ratio (SNR). The purpose of Wiener filtering is to minimize the MSE, $E\{[\hat{S}(t) - S(t)]^2\}$, between our estimate $\hat{S}(t)$ (i.e. the output of the filter) and the actual signal $S(t)$. Since the output of the filter can be written as $S(t) + (\hat{S}(t) - S(t))$, we use the signal power divided by the MSE as an output SNR. For the purpose of our graphs we translate this to dB. The horizontal axis is $10 \log_{10}(10 v_S^2/v_N^2)$, the input SNR in dB.

The top line in Fig. 1 gives the performance of the Wiener filter H_o^*, designed using σ_o and ν_o of (4) in equation (2), when σ_o and ν_o are, in fact, the signal and noise spectra which occur. For this case it is straightforward, via equation (3), to show that

$$e^*(\sigma_o, \nu_o)/v_S^2 = \alpha_S \alpha_N / \sqrt{(\alpha_S \alpha_N^2 r + \alpha_S^2 \alpha_N)(\alpha_S r + \alpha_N)}$$

where

$$r \triangleq v_S^2/v_N^2 .$$

Now, suppose that the only information about which we are certain is

the powers of the signal v_S^2 and the noise v_N^2 and that we have estimated with sufficient accuracy the fractional power of each on the set $S = \{\omega \text{ real} : |\omega| \leq 1\}$. We denote the signal and noise fractional powers by p_S and p_N, respectively (e.g. $(2\pi)^{-1} \int_S \sigma(\omega) d\omega = p_S v_S^2$). In particular, for the example considered above, we have $p_S = .5$ and $p_N = .063$. If these total powers and fractional powers are all we can really be certain of, we would like to know how badly the performance of H_o^* can deteriorate. The bottom line in Fig. 1 gives the worst-case performance of H_o^*. The middle line represents what we can do trivially for any pair of spectra by using an all-pass filter ($H \equiv 1$) when the input SNR is positive and by using a no-pass filter ($H \equiv 0$) when the input SNR is negative. Thus we see that if the spectra are actually first-order Markov then our filter does well, but if not we can do *significantly worse than trivial filtering*.

Finally we note that uncertainty classes of spectra given by assuming exact knowledge only of the total and fractional powers are called p-point classes and have been studied as models of spectral uncertainty by Cimini and Kassam (1980). An analogous uncertainty class for probabilities used in robust hypothesis testing and robust detection has been examined by the authors (Vastola and Poor, 1980) and by El-Sawy and VandeLinde (1977, 1979).

The ε-contamination class. Suppose that we again have a particular spectral pair (σ_o, ν_o) which we believe to be the true spectra, but that we also have a general sense of uncertainty about our choice which we model by an ε-contaminated class; i.e., we assume we know that the true spectra satisfy $(\sigma, \nu) \in \mathcal{S}_\varepsilon \times \mathcal{N}_\varepsilon$ where $0 \leq \varepsilon \leq 1$,

$$\mathscr{S}_\varepsilon = \{\sigma \mid \sigma(\omega) = (1-\varepsilon)\sigma_o(\omega) + \varepsilon\sigma'(\omega) \quad \omega\in\mathbb{R}, \int_{-\infty}^{\infty}\sigma'(\omega)\,d\omega = \int_{-\infty}^{\infty}\sigma_o(\omega)\,d\omega\}$$

and

$$\mathscr{N}_\varepsilon = \{\nu \mid \nu(\omega) = (1-\varepsilon)\nu_o(\omega) + \varepsilon\nu'(\omega) \quad \omega\in\mathbb{R}, \int_{-\infty}^{\infty}\nu'(\omega)\,d\omega = \int_{-\infty}^{\infty}\nu_o(\omega)\,d\omega\} \quad .$$

Classes of this form have been used extensively as general models of uncertainty (Tukey, 1960; Huber, 1965; Kassam and Lim, 1977; Hosoya, 1978).

Fig. 2 gives the performance of the Wiener filter H_o^* designed via equation (2) assuming a narrow-band ($\alpha_S = 1$) first-order Markov signal in wide-band ($\alpha_N = 1000$) first-order Markov noise. The upper line gives the performance of this filter when these are the true signal and noise. The lower line is the worst case of this filter over the uncertainty classes in (5) with σ_o and ν_o given by the above choices and with $\varepsilon = .1$. We see that, for values of input SNR near zero, the worst case is better than trivial filtering but still much worse than optimal (about 8.5 dB); for values of input SNR greater in absolute value than 60 the performance in both the nominal and worst cases is the same as trivial filtering; and for all other values the worst case is worse than trivial filtering.

An ε-contaminated signal in white noise. Fig. 3 shows the nominal and worst case performance of the nominal Wiener filter for the signal uncertainty class $\mathscr{S}_\varepsilon$ in (5) with $\varepsilon = .1$ and σ_o first-order Markov with $\alpha_S = 1$. The noise is white noise with no uncertainty and the horizontal axis is actually the ratio of signal power v_S^2 to the noise level $N_o/2$. Note that the worst case is bounded above by 10; in fact, for any choice of ε, it is bounded above by $10 \log(10\varepsilon)$.

As noted above, the optimal and worst-case performance of Wiener

filtering under various conditions has been examined extensively for several uncertainty models and for a variety of signal and noise parameters (such as bandwidth and power). The above examples are representative of the sensitivity of Wiener filtering to deviations from spectral assumptions which were found in virtually every case.

3. Robust Wiener Filters

To remedy the problems of Wiener filtering sensitivity discussed in the preceding section, we consider the following robust filter design which was developed by Poor (1980) based on the work of Kassam and Lim (1977).

A most-robust Wiener filter (Poor, 1980) is a solution H_R to the game

$$\min_{H} \sup_{(\sigma,\nu)\in \mathcal{S}\times\mathcal{N}} e(\sigma,\nu;H) \tag{6}$$

where $\mathcal{S}$ and $\mathcal{N}$ are classes of spectra representing uncertainty in the signal and noise, respectively, and where $e(\sigma,\nu;H)$ is given in (1). Note that since the sup in (6) gives the least upper bound on the error, H_R^* is a filter with the smallest possible such upper bound. In other words H_R^* is least sensitive to worst case uncertainty.

A pair of spectra (σ_L,ν_L) is least favorable for Wiener filtering for the spectral uncertainty classes $\mathcal{S}$ and $\mathcal{N}$ (Poor, 1980) if

$$e(\sigma,\nu;H_L^*) \leq e(\sigma_L,\nu_L;H_L^*) \tag{7}$$

for all $\sigma \in \mathcal{S}$, $\nu \in \mathcal{N}$ where H_L^* is the Wiener filter for the pair (σ_L,ν_L) as in (2).

It is straightforward to see that if $(\sigma_L,\nu_L) \in \mathcal{S}\times\mathcal{N}$ is least favorable

for Wiener filtering for $\mathscr{S}$ and $\mathscr{N}$ then the pair $((\sigma_L,\nu_L),H_L^*)$ is a saddle-point solution to the minimax game (6). That is,

$$\sup_{(\sigma,\nu)\in\mathscr{S}\times\mathscr{N}} e(\sigma,\nu;H_L^*) = e(\sigma_L,\nu_L;H_L^*) = \min_H e(\sigma_L,\nu_L;H). \qquad (8)$$

We see from this that if (σ_L,ν_L) is least favorable then H_L^* is a most-robust Wiener filter.

Thus we see that if we can find a least-favorable pair then we can design a most-robust Wiener filter. One of the methods developed in (Poor, 1980) for finding least favorable pairs of spectra (and hence most-robust filters) involves an analogous concept in hypothesis testing: least-favorable probability density functions (PDF's) for testing one set of PDF's against another. Least-favorable PDF's have been found for a variety of classes of PDF's (Huber, 1965; Kassam, 1981; Vastola and Poor, 1980). If every signal spectrum in $\mathscr{S}$ has the same finite power v_S^2 and every noise spectrum in $\mathscr{N}$ has the same finite power v_N^2 then we can define classes of PDF's

$$\mathscr{P}_S = \{f_S \mid f_S(\omega) = \sigma(\omega)/2\pi v_S^2,\ \sigma\in\mathscr{S}\}$$

and

$$\mathscr{P}_N = \{f_N \mid f_N(\omega) = \nu(\omega)/2\pi v_N^2,\ \nu\in\mathscr{N}\}$$

and possibly apply the following (Poor, 1980, Corollary 1).

Theorem. *If $\mathscr{S}$ and $\mathscr{N}$ are convex and have constant powers v_S^2 and v_N^2, respectively, and $q_S \in \mathscr{P}_S$ and $q_N \in \mathscr{P}_N$ are least-favorable PDF's for $\mathscr{P}_S$ versus $\mathscr{P}_N$ then $\sigma_L \triangleq 2\pi v_S^2 q_S$ and $\nu_L \triangleq 2\pi v_N^2 q_N$ are least favorable spectra for Wiener filtering for $\mathscr{S}$ and $\mathscr{N}$.*

This theorem allows us to construct most-robust Wiener filters for the

first two examples considered in Section 2.

The p-point class. It can be seen from the above theorem and Vastola and Poor (1980) or from Cimini and Kassam (1980) that

$$H_R^*(\omega) = \begin{cases} \dfrac{p_S v_S^2}{p_S v_S^2 + p_N v_N^2} & \text{for } \omega \in S \\[2ex] \dfrac{(1-p_S)v_S^2}{(1-p_S)v_S^2 + (1-p_N)v_N^2} & \text{for } \omega \in S^c \end{cases}$$

and, hence,

$$e(\sigma,\nu;H_R^*) = \frac{p_S p_N}{p_S r + p_N} + \frac{(1-p_S)(1-p_N)}{(1-p_S)r + (1-p_N)} \text{ for all } (\sigma,\nu) \in \mathcal{S}\times\mathcal{N},$$

where $r \triangleq v_S^2/v_N^2$, the input SNR. In Fig. 4 we have superimposed onto Fig. 1 the performance of H_R^* (the middle line). It is clear from Fig. 4 that, unless we are extremely certain about our choice of σ and ν, H_R^* is preferable to H_o^*.

The ε-contaminated class. For the classes in (4) it can be easily seen from the above theorem and Huber (1965) that

$$H_R^*(\omega) = \begin{cases} k' \triangleq c'r/(c'r+1) & \text{for } H_o^*(\omega) \le k' \\ H_o^*(\omega) & \text{for } k' < H_o^*(\omega) < k'' \\ k'' \triangleq c''r/(c''r+1) & \text{for } H_o^*(\omega) \ge k'' , \end{cases}$$

where $0 \le c' < c'' \le \infty$ are constants given in Huber (1965). It is interesting to note that the robust filter H_R^* has this same form for several other

uncertainty models (Poor, 1980; Huber, 1977). Also note that this H_R^* will not have constant MSE over $\mathcal{S} \times \mathcal{N}$ as in the previous example. In Fig. 5 we have superimposed onto Fig. 2 the performance of H_R^* when the true spectral pair is (σ_o, ν_o) (the second line from the top) and when the true spectral pair is (σ_L, ν_L) (the third line from the top). Recall from the definition of (σ_L, ν_L) that the latter is the worst-case performance of H_R^*. For this example $c' = 1/c'' = .125$.

Unlike the preceding example, the preferability of the most-robust filter is not so clear-cut. If one were relatively certain about (σ_0, ν_0) being correct then H_0^* would be the better choice; however, if not, and if the guaranteed level of performance over $\mathcal{S} \times \mathcal{N}$ (given by the third line down) were adequate, we would likely choose H_R^*.

An ε-contaminated signal in white noise. Clearly the above theorem cannot be applied to find a robust filter in this case since the noise has infinite power; however a more direct approach proves fruitful here. First, we may restrict our search to $H \in L_2(d\omega)$, the mean-square integrable functions on $\mathbb{R}$, the real line, since all others have infinite MSE regardless of what σ is (cf. equation (1)). Second, we have, for all $H \in L_2(d\omega)$,

$$\sup_{\sigma \in \mathcal{S}_\varepsilon} e(\sigma, \nu_o; H) = \sup_{\sigma \in \mathcal{S}_\varepsilon} \frac{1}{2\pi} \int_{\mathbb{R}} [|1-H(\omega)|^2 ((1-\varepsilon)\sigma_o(\omega) + \varepsilon\sigma'(\omega)) + |H(\omega)|^2 \frac{N_o}{2}] \, d\omega$$

$$= e((1-\varepsilon)\sigma_o, \nu_o; H) + \varepsilon \sup_{\sigma'} \frac{1}{2\pi} \int_{\mathbb{R}} |1-H(\omega)|^2 \sigma'(\omega) \, d\omega$$

$$= e((1-\varepsilon)\sigma_o, \nu_o; H) + \varepsilon v_S^2$$

The last step is true because σ' is assumed to be integrable and $H \in L_2(d\omega)$.

Clearly this, equation (2), and (6) (the definition of H_R^*) imply that

$$H_R^*(\omega) = \frac{(1-\varepsilon)\sigma_o(\omega)}{(1-\varepsilon)\sigma_o(\omega) + N_o/2} .$$

Recall that Fig. 3 showed the performance of H_o^* in this situation in its nominal and worst cases. If we superimposed the nominal and worst cases of H_R^* onto Fig. 3, as we have done for the other examples, we would find no change; i.e., up to the accuracy of the graph the nominal cases and worst cases of H_o^* and H_R^* are the same. In fact, they differ by no more than .01. It should be noted that this is a singular example and the unusual performance is due to the infinite power of the white noise, not to the "very wide bandedness" which white noise is generally used to model.

IV. *Discussion and Conclusions*

As we have discussed above, the results presented in this paper (with the one exception of the white noise example) are representative of our findings in a wide variety of cases. For example, although it is a much harder case to solve, we have developed numerical results for causal Wiener filtering of an ε-contaminated first-order Markov signal in first-order Markov noise. The theory of the causal case has not been developed in the same generality as the noncausal case; however, this specific example can be treated using the results of Poor (1980) and Yao (1971). In Fig. 6 we have presented the results for this causal filtering example with $\varepsilon = .1$, $\alpha_S = 1$ and $\alpha_N = 1000$. For comparison we have also included Fig. 7 which gives the

results for the corresponding noncausal case. Note the similarity between the two figures. Again, this is indicative of our findings over a wide range of signal and noise bandwidths and ϵ's.

Other situations we examined in the noncausal case include ones with σ and/or ν as second-order Markov (i.e. having the form $4\alpha^3 v^2/(\alpha^2 + \omega^2)^2$) or using bandlimited white noise. The results for all these cases were very similar to those already presented (e.g. Fig. 5).

Of particular interest is the case of an ϵ-contaminated first-order Markov signal in ϵ-contaminated bandlimited white noise. Even when the bandwidth of the noise was extremely large (e.g. 10^6) the results were similar to the other cases and unlike those involving nonbandlimited white noise (cf. the remarks at the end of Section 3).

In summary, the Wiener filter can be undesirably sensitive to small deviations from assumed spectral models. Furthermore, while there are enough specific cases to the contrary to make caution advisable, we have found for a wide variety of situations that, when spectral uncertainty exists, the robust Wiener filter is generally preferable to the traditional Wiener filter.

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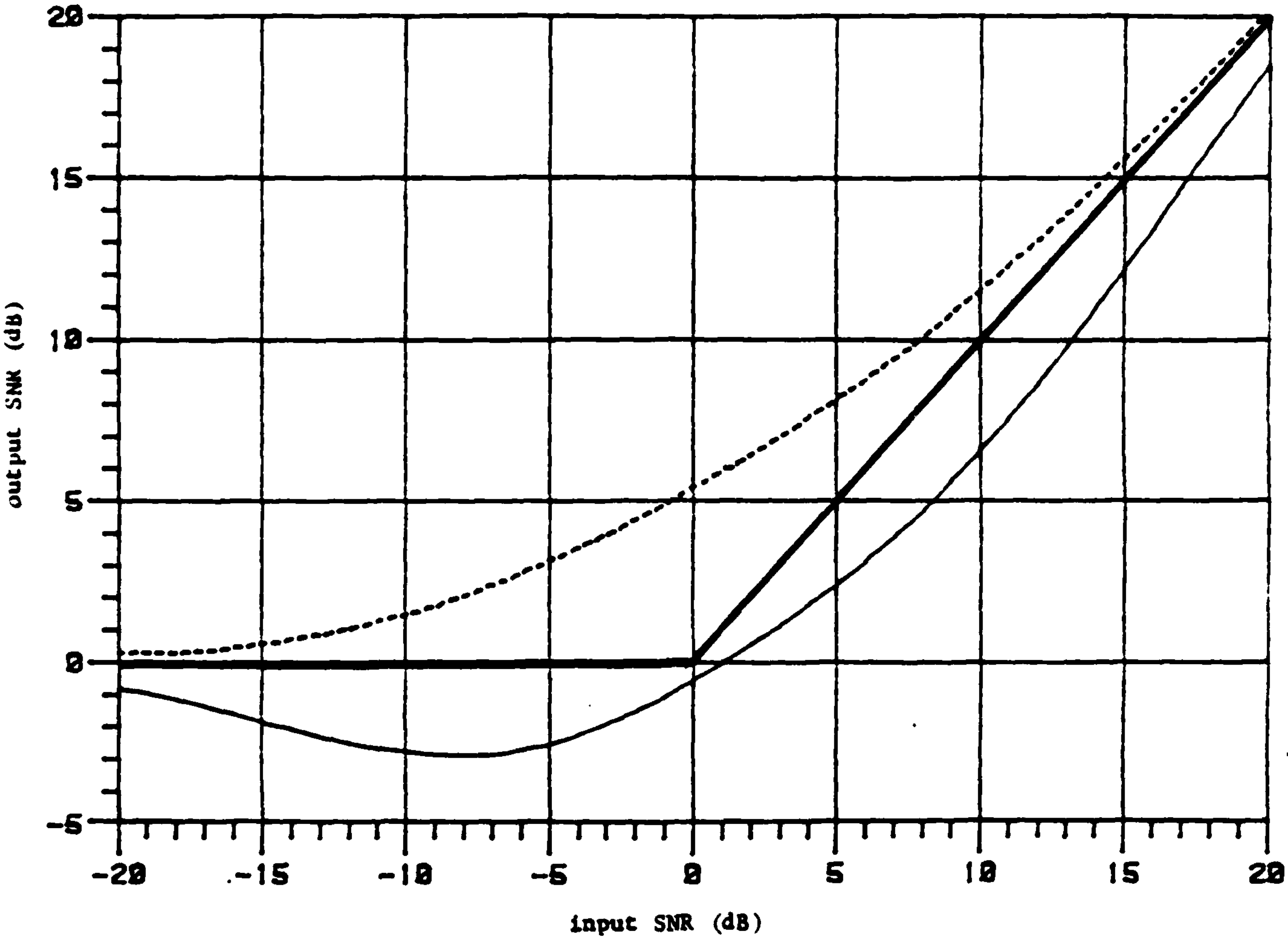

FIG. 1. p-point example. (from top to bottom) H_0^* at (σ_0, ν_0); "trivial filtering"; H_0^* at its worst case.

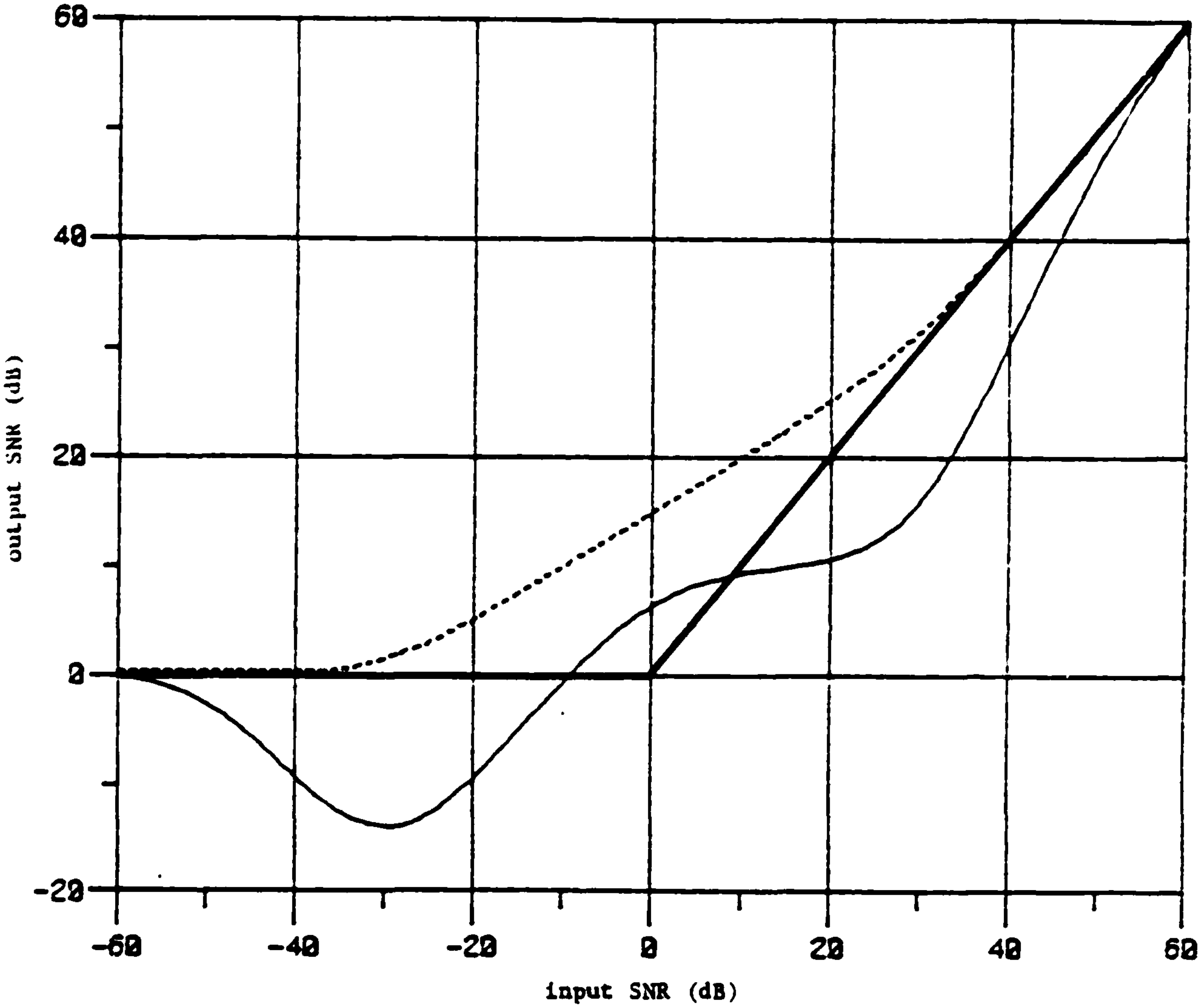

FIG. 2. ε-contaminated example. H_0^* at (σ_0, ν_0) and at its worst case.

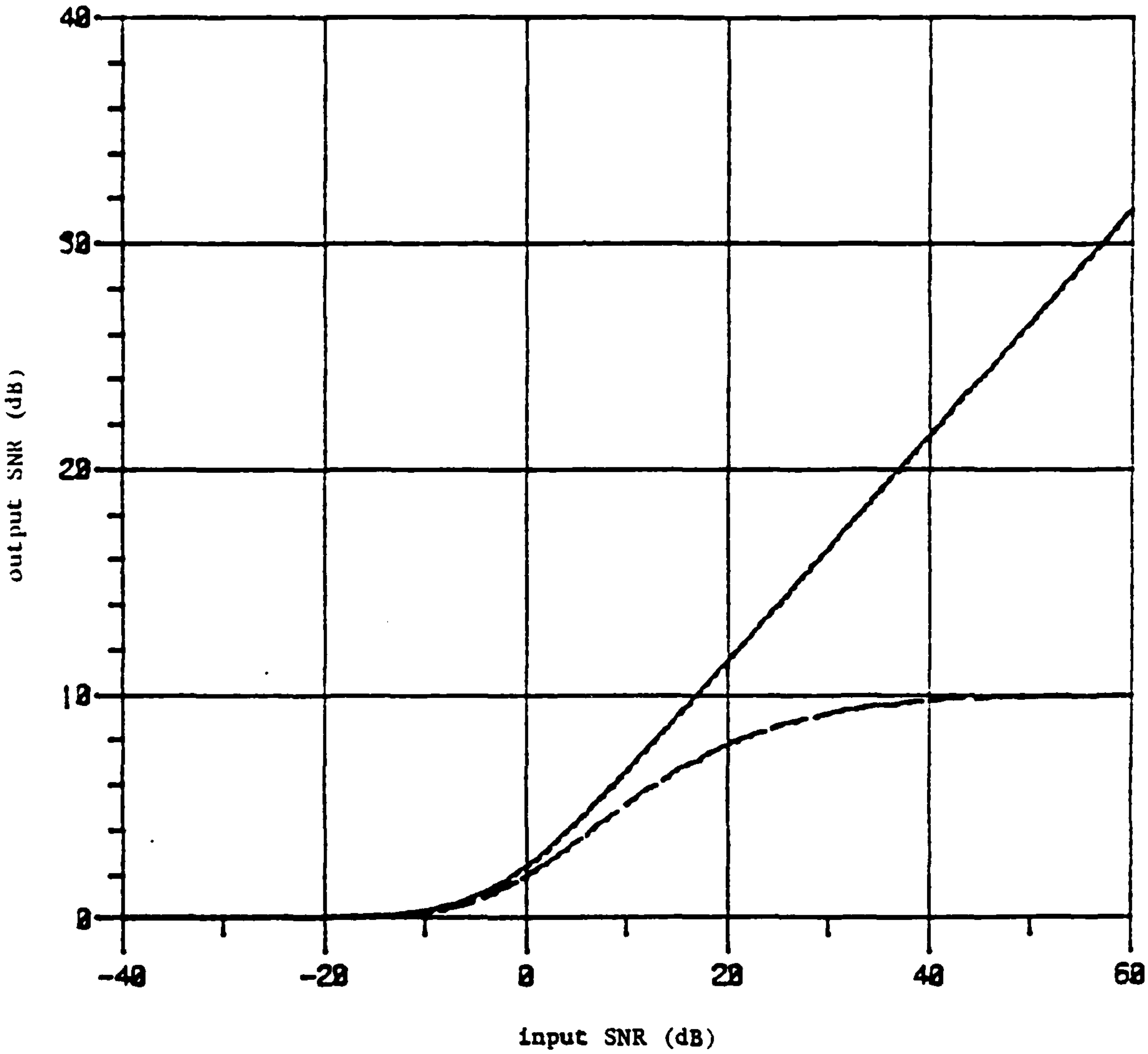

FIG. 3. ε-contaminated signal in white noise. H_0^* at (σ_0,ν_0) and at its worst case.

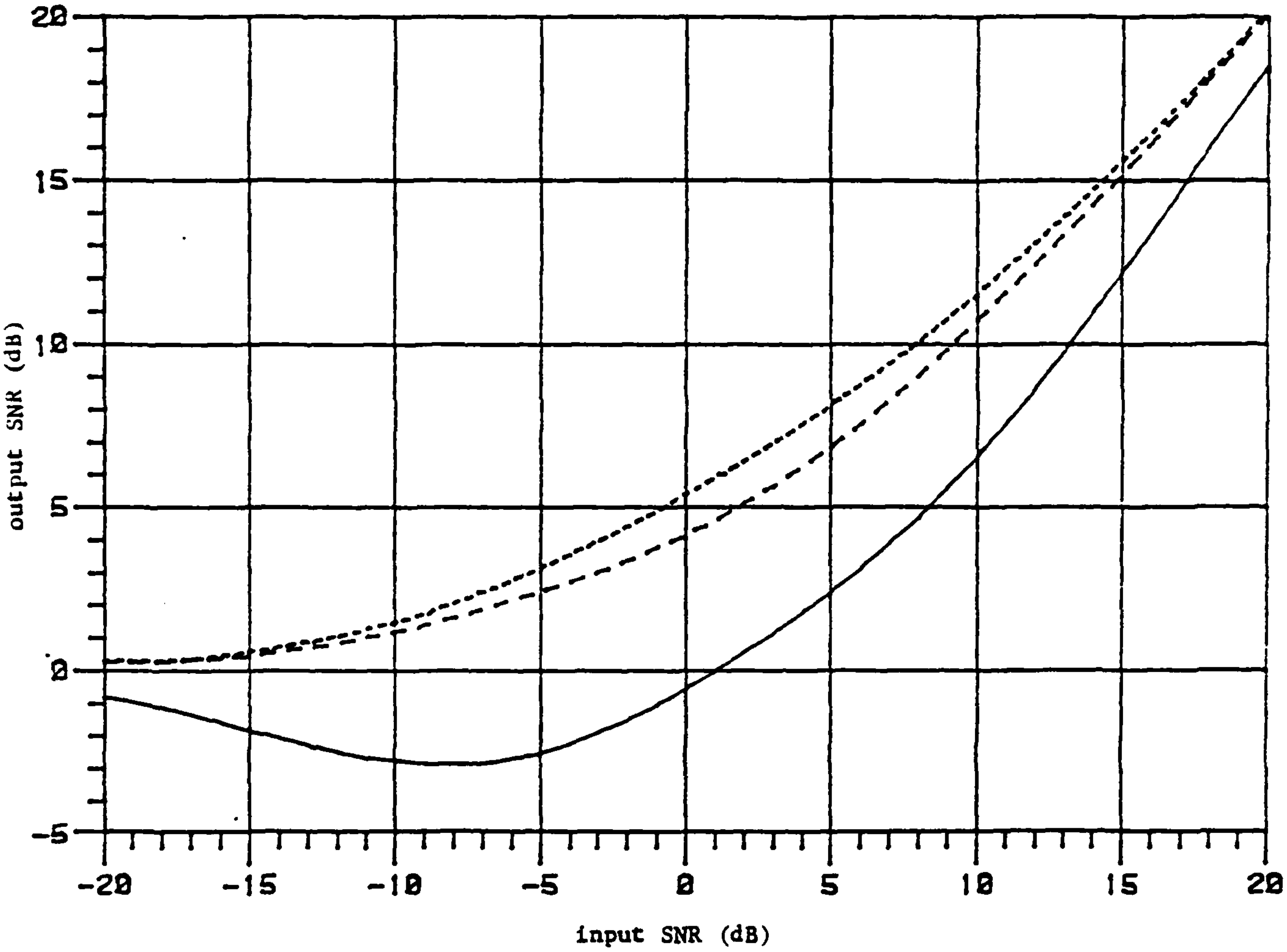

FIG. 4. p-point example. (from top to bottom) H_0^* at (σ_0,ν_0); H_R^* at any $(\sigma,\nu) \in \mathcal{S} \times \mathcal{N}$; H_0^* at its worst case.

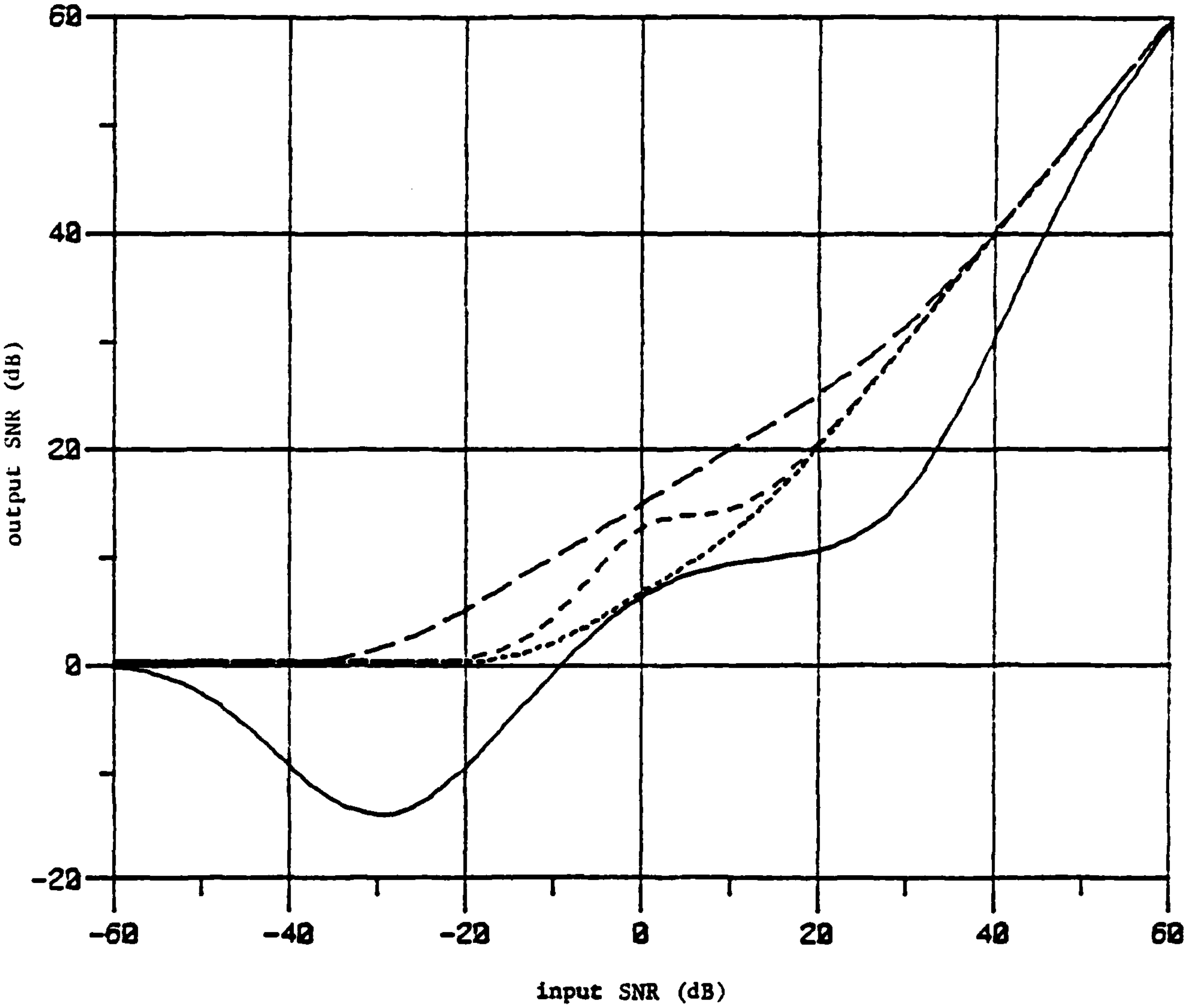

FIG. 5. ε-contaminated example. (from top to bottom) H_0^* at (σ_0,ν_0); H_R^* at (σ_0,ν_0); H_R^* at (σ_L,ν_L) (H_R^*'s worst case); H_0^* at its worst case.

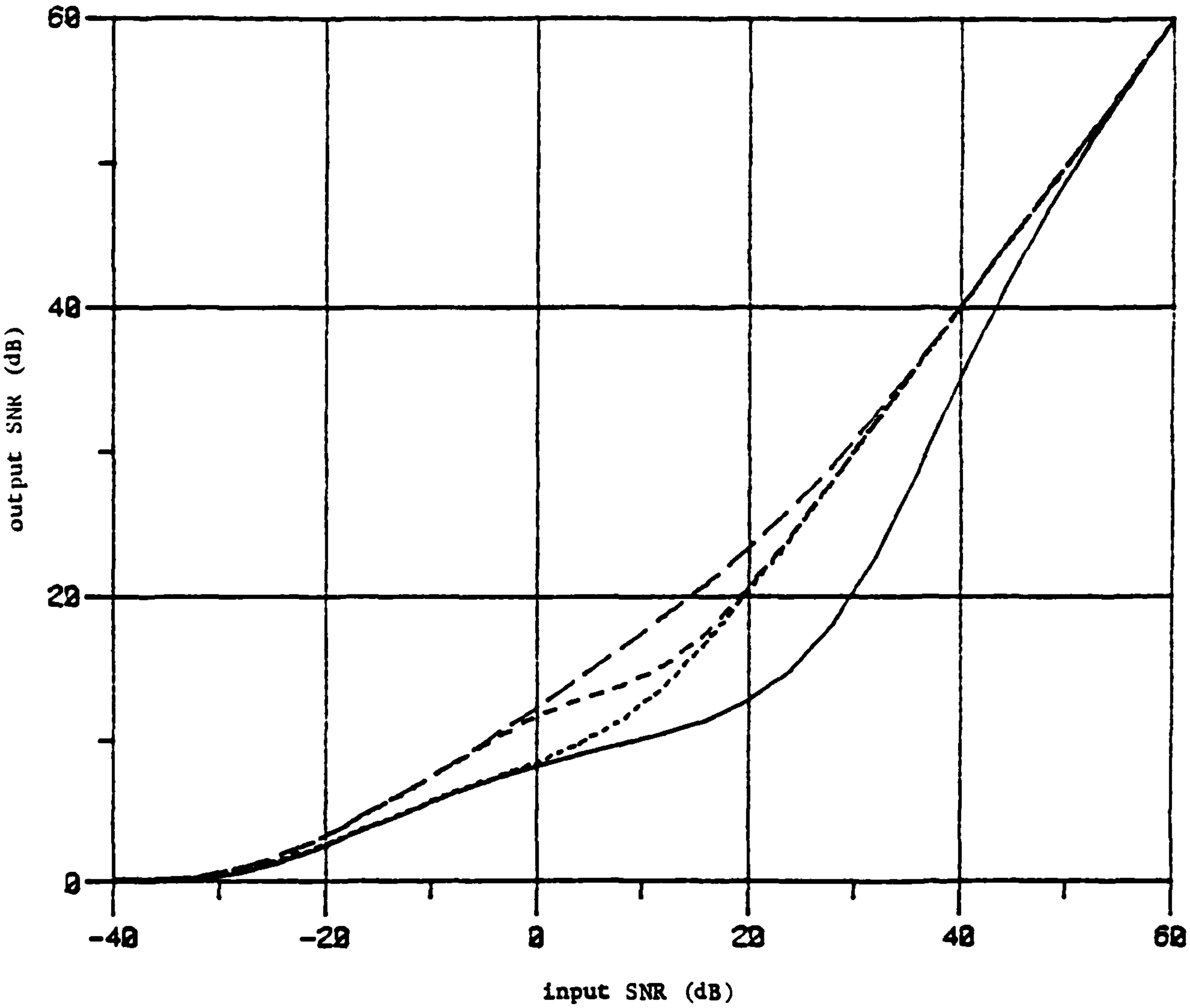

FIG. 6. Causal example. (from top to bottom) H_0^* at (σ_0, ν_0); H_R^* at (σ_0, ν_0); H_R^* at (σ_L, ν_L) (H_R^*'s worst case); H_0^* at its worst case.

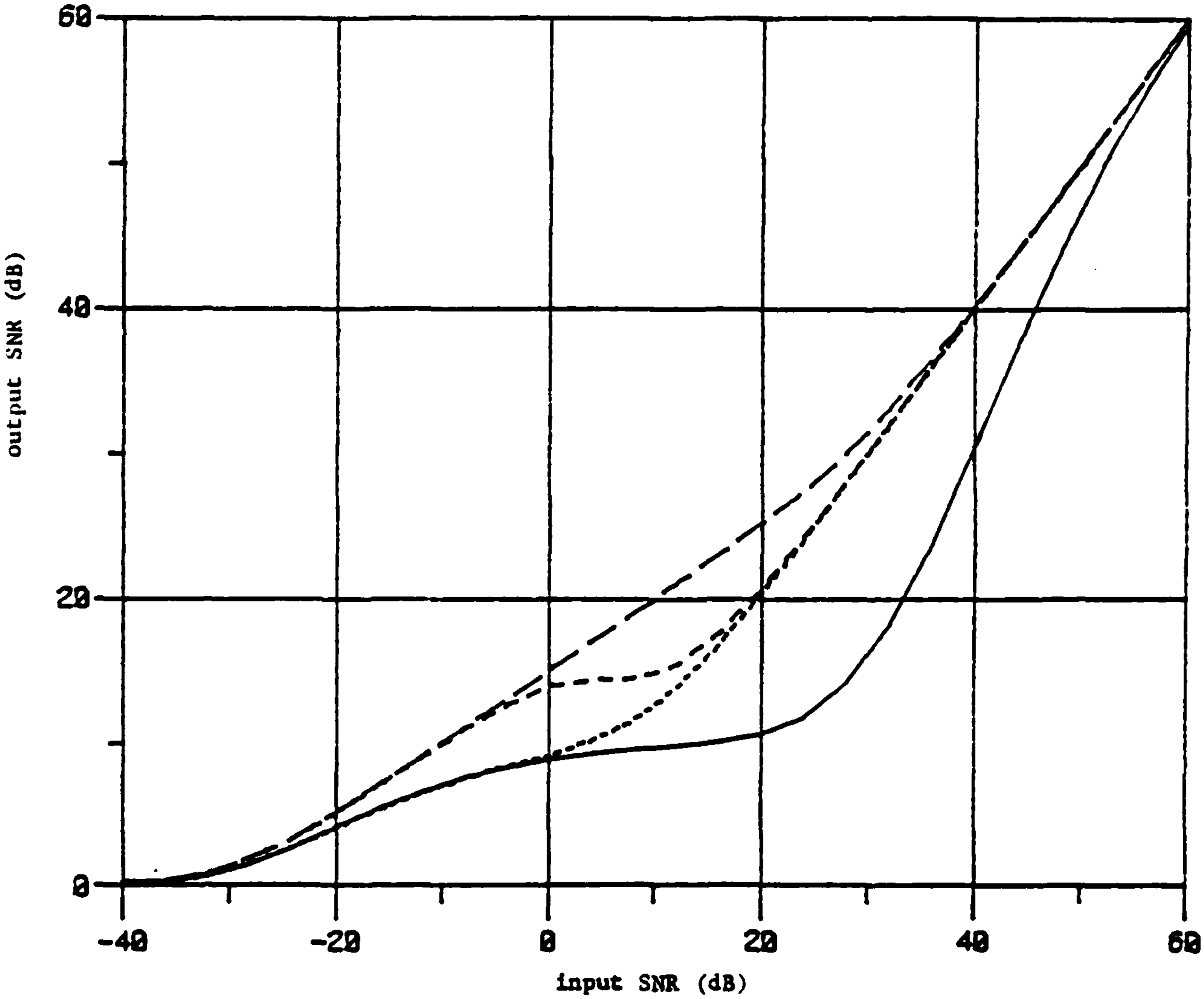

FIG. 7. Noncausal version of FIG. 6 (all parameters the same).

[9] K. S. Vastola and H. V. Poor, "Robust Linear Estimation of Stationary Discrete-Time Signals," Proceedings of the 1981 Conference on Information Sciences and Systems, The Johns Hopkins University, Baltimore, MD, March 1981, pp. 512-516.

ROBUST LINEAR ESTIMATION OF STATIONARY DISCRETE-TIME SIGNALS

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ABSTRACT

A minimax formulation of the problem of designing discrete-time smoothers, filters and predictors when knowledge of the signal and noise spectra is inexact is presented. A result is given which converts the minimax problem to a maximization problem. Explicit solutions are given for the case of a contaminated wide-sense Markov signal in white noise. The results of a numerical study of the inherent trade-off involved in robust filtering are given for this case and the preferability of the robust filter is demonstrated for a wide range of input SNR's and signal bandwidths.

1. INTRODUCTION

In the traditional formulation of the linear minimum-mean-square-error (MMSE) signal estimation problem, if the signal and noise processes are uncorrelated with each other, the solutions to both the causal and noncausal versions of this problem depend only on the power spectra of the signal and noise. In practice, however, the spectral properties of the signal and/or noise may not be known with complete certainty. To account for such uncertainty several studies have investigated possible designs of robust filters. Robust filters are filters which, while not necessarily performing as well as the optimal filter when the signal and noise spectra are as anticipated, do not experience a significant decrease in relative performance over some class of spectra "near" the anticipated pair. On the other hand the so-called optimal filter (i.e., the one designed based on the anticipated model) may experience dramatic degradation of performance under small deviations (see [5]).

So far the thrust of this research has been for continuous-time filtering ([1],[2]). In this paper we consider the discrete-time case for causal linear signal estimation (which includes smoothing, filtering, and prediction). In Section 2 we present a formulation of the robust linear causal discrete-time signal estimation problem and a minimax type theorem which yields a general design approach for these robust signal estimators. This formulation is analogous to that presented in Section II of [2] for continuous-time robust filtering. In Section 3 we consider specific models of signal and noise and present some numerical results. Finally in Section 4 we discuss these results along with some possible topics for further study in this area.

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2. THE GENERAL FORMULATION

Throughout this paper we denote the set of integers by Z and we assume that we observe a realization $\{y(k)|k\in Z\}$ of a random process $\{Y(k)|k\in Z\}$ given by

$$Y(k) = S(k) + N(k), \qquad k \in Z, \tag{2.1}$$

where $\{S(k)|k\in Z\}$ and $\{N(k)|k\in Z\}$ are wide-sense stationary random processes which are uncorrelated with each other. We denote their spectral measures by m_S and m_N, respectively. We also assume that the mean of $\{N(k)|k\in Z\}$ is zero. Our purpose is to form a linear estimate of a linear function of $\{s(k)\}$ (the signal) from $\{y(k)\}$ (the observation).

By a spectral measure we mean specifically a nonnegative Borel measure (i.e. a countably additive measure on the Borel sets, see [4]) on U, the unit circle of the complex plane. Given such a spectral measure m we denote by $H_+^2(dm)$ the Hardy subspace of $L^2(dm)$, which is the subspace spanned by $\{e^{in\theta}|n = 0,1,2,\ldots\}$ [4]. For the estimation problem discussed above $H_+^2(d(m_S+m_N))$ is the space of all mean-square integrable <u>causal</u> transfer functions, while $L^2(d(m_S+m_N))$ also includes the noncausal ones.

Let $D(\theta)$ represent a "desired" linear operation on the signal process $\{S(k)\}$; for example $D(\theta) = e^{-in\theta}$ represents smoothing for $n < 0$, filtering for $n = 0$ and prediction for $n > 0$. For each pair of spectral measures (m_S,m_N) and transfer function $H \in L^2(d(m_S+m_N))$ the mean-square error $e_D(dm_S,dm_N;H)$ is given by

$$(2\pi)^{-1}\int_{-\pi}^{\pi}|D(\theta)-H(\theta)|^2\, dm_S(\theta) + (2\pi)^{-1}\int_{-\pi}^{\pi}|H(\theta)|^2\, dm_N(\theta)\ . \tag{2.2}$$

For each pair (m_S,m_N) the problem

$$\min_{H \in H_+^2(d(m_S+m_N))} e_D(dm_S,dm_N;H) \tag{2.3}$$

has a solution $H^\dagger$ (the optimal causal transfer function) which is uniquely defined a.e. with

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respect to $m_S + m_N$. We denote by $e_D^+(dm_S, dm_N)$ the optimal value $e_D(dm_S, dm_N; H^+)$.

Unfortunately, there is no general expression for $e_D^+(dm_S, dm_N)$, however for a wide variety of cases (all when D and the derivative with respect to Lebesgue measure of the absolutely continuous part of m_N are rational) Snyders [3] has given specific expressions; further he has given a systematic method of finding others. This will be important in Section 3.

Of course, this traditional formulation of discrete-time causal signal estimation assumes exact knowledge of the signal and noise spectra. As we mentioned in Section 1 it is often unrealistic to assume this knowledge, and hence it is of interest to design a robust signal estimator. The design approach we consider here is analogous to that of robust Wiener filtering as considered by Poor [2] in a formulation based on the work of Kassam and Lim [1].

Specifically, we assume that the signal and noise spectra, m_S and m_N, are known only to belong to some nonintersecting classes, $\mathcal{J}$ and $\mathcal{N}$. That is, we know that $m_S \in \mathcal{J}$ and $m_N \in \mathcal{N}$ but we do not know which members of $\mathcal{J}$ and $\mathcal{N}$ are the true signal and noise spectra.

We consider a transfer function H_R to be robust for $\mathcal{J}$ and $\mathcal{N}$ if there is an upper bound for $e(m_S, m_N; H_R)$ over all $(m_S, m_N) \in \mathcal{J} \times \mathcal{N}$. The adjective robust is appropriate since this upper bound would give a guaranteed level of performance. Ideally we would like to have the best such transfer function. Thus we define a most robust causal transfer function to be a solution H_R^+ to the game

$$\min_{H \in K^+} \sup_{(m_S, m_N) \in \mathcal{J} \times \mathcal{N}} e_D(dm_S, dm_N; H) \tag{2.4}$$

where $K^+ = \{H \in H_+^2(d\theta) \mid \sup_{\theta \in U} |H(\theta)| \le 1\}$.

In an effort to find a most robust causal transfer function we make the following definition.

Definition: A pair of spectral measures (m_S^L, m_N^L) is least favorable for causal linear estimation for the classes $\mathcal{J}$ and $\mathcal{N}$ if

$$e_D(dm_S, dm_N; H_L^+) \le e_D(dm_S^L, dm_N^L; H_L^+) \tag{2.5}$$

for all $(m_S, m_N) \in \mathcal{J} \times \mathcal{N}$, where H_L^+ is the optimal causal transfer function for (m_S^L, m_N^L) defined via (2.3).

It is straightforward to see that a pair $(m_S^L, m_N^L) \in \mathcal{J} \times \mathcal{N}$ is least favorable if and only if it and its optimal transfer function form a saddlepoint solution to the game (2.4), i.e.

$$\min_{H \in K^+} e_D(dm_S^L, dm_N^L; H) = e_D(dm_S^L, dm_N^L; H_L^+)$$
$$= \sup_{(m_S, m_N) \in \mathcal{J} \times \mathcal{N}} e_D(dm_S, dm_N; H_L^+) \tag{2.6}$$

Thus if we could find a least-favorable pair $(m_S, m_N) \in \mathcal{J} \times \mathcal{N}$, we would have our sought-after most-robust estimator: the optimal causal transfer function for (m_S, m_N). The purpose of our main theorem (below) is to facilitate the search for a least-favorable pair.

We now view $\mathcal{J}$ and $\mathcal{N}$ as subsets of the Banach space, $\mathcal{B}$, of finite Borel measures on U and consider $\mathcal{B}$ endowed with its weak* topology (see [10]). A sequence $\{m_n\}$ converges to a measure m in this topology if $\int_U f\, dm_n$ converges to $\int_U f\, dm$ for every continuous function f on U. (In the probability literature this is usually referred to as weak convergence of (probability) measures (see [9]).)

We are now ready to present our main theorem which reduces the problem of finding a least-favorable pair (and hence a robust transfer function) to maximizing the functional $e^+(dm_S, dm_N)$ over $\mathcal{J} \times \mathcal{N}$. This is accomplished by converting the minimax problem (2.4) to a maximin problem whose "min" part (given by (2.3)) has already been solved by Snyders [3].

Theorem. If $\mathcal{J}$ and $\mathcal{N}$ are nonintersecting convex weak*-compact subsets of $\mathcal{B}$ then $(m_S^L, m_N^L) \in \mathcal{J} \times \mathcal{N}$ is a least-favorable pair for causal linear estimation if and only if

$$e_D^+(dm_S^L, dm_N^L) = \max_{(m_S, m_N) \in \mathcal{J} \times \mathcal{N}} e_D^+(dm_S, dm_N) \tag{2.7}$$

where $e_D^+(dm_S, dm_N)$ is defined above via (2.3).

Proof. If "only if" part of the theorem follows directly from the definitions of least-favorability and $e_D^+(dm_S, dm_N)$; i.e., for all $(m_S, m_N) \in \mathcal{J} \times \mathcal{N}$, $e_D^+(dm_S^L, dm_N^L) \ge e_D(dm_S, dm_N; H_L^+) \ge e_D^+(dm_S, dm_N)$. In fact this is so for arbitrary subsets, $\mathcal{J}$ and $\mathcal{N}$, of $\mathcal{B}$. On the other hand, the "if" part does depend on the hypothesis.

By definition a pair (m_S^L, m_N^L) is least favorable for causal linear estimation for classes $\mathcal{J}$ and $\mathcal{N}$ of spectra if $e_D(dm_S, dm_N; H_L^+) \le e_D(dm_S^L, dm_N^L; H_L^+)$ for all $(m_S, m_N) \in \mathcal{J} \times \mathcal{N}$ where H_L^+ is the optimal causal transfer function for (m_S^L, m_N^L) defined by (2.3). We wish to show that this is true for a pair $(m_S^L, m_N^L) \in \mathcal{J} \times \mathcal{N}$ if $\mathcal{J}$ and $\mathcal{N}$ are weak* compact and if (2.7) holds, i.e. if $e_D(dm_S^L, dm_N^L; H_L^+) \ge e_D(dm_S, dm_N; H^+)$, for all $(m_S, m_N) \in \mathcal{J} \times \mathcal{N}$, where, for each pair (m_S, m_N), H^+ is the

causal transfer function which is optimal for that pair.

Note that if minimax equality holds for the game (2.4) (i.e. if (2.6) holds) then the pair (m_S^L, m_N^L) satisfying (2.7) is clearly least favorable for $\mathcal{S}$ and $\mathcal{N}$ (see Equation (2.6)). Thus our problem reduces to showing that minimax equality holds. We will make use of the following (from Theorem 3.5, pp. 143-144 in [11]).

<u>Lemma</u>. If X and Y are separable topological linear spaces; A and B are compact convex subsets of X and Y, respectively; and F is a real valued upper-lower-semicontinuous concave-convex function on A × B then F satisfies the minimax equality on A × B.

We wish to apply this lemma with $X = \mathcal{M} \times \mathcal{M}$ (recall that $\mathcal{M}$ is the space of finite Borel measures on $(U,\mathcal{A})$ endowed with its weak* topology, X is endowed with the corresponding product topology); $Y = H_+^2(d\theta)$ (the Hardy subspace of $L^2(d\theta)$ spanned by $\{e^{in\theta} \mid n = 0,1,2,\ldots\}$ here endowed with its weak topology); $A = \mathcal{S} \times \mathcal{N}$; and $B = K^\dagger$. Our function F on A × B is given by $F((m_S,m_N),H) = e_D(dm_S,dm_N;H)$.

The only property which is not obvious (or at least straightforward to show) is that $K^\dagger$ is compact. From [10], Sections V.3 and V.4, we see that all we need to show is that $K^\dagger$ is *bounded and closed in the norm topology of* $H_+^2(d\theta)$.

Boundedness is straightforward since $\|H\|_2^2 = \int_{-\pi}^{\pi} |H(\theta)|^2 d\theta \le 2\pi$. For closedness we must show that if $H_n \to H$ in norm (i.e. if $\lim_{n\to\infty} \int_{-\pi}^{\pi} |H_n(\theta) - H(\theta)|^2 d\theta = 0$) and if $\{H_n\} \subseteq K^\dagger$ then $H \in K^\dagger$. Assume $H \notin K^\dagger$ then there is a Borel set $E \subseteq U$ such that $\lambda(E) > 0$ (where λ is Lebesgue-Borel measure on U) and $H(\theta) > 1 + \epsilon$ for some $\epsilon > 0$ and for all $\theta \in E$. But $\int_{-\pi}^{\pi} |H_n(\theta) - H(\theta)|^2 d\theta \ge \int_E |H_n(\theta) - H(\theta)|^2 d\theta > \epsilon^2 \lambda(E) > 0$ for all n. Contradiction! Hence $K^\dagger$ is compact in the weak topology of $H_+^2(d\theta)$ and we may apply the Lemma. This completes the proof of the theorem.

The hypothesis of the theorem (that $\mathcal{S}$ and $\mathcal{N}$ *be convex weak*-compact subsets of* $\mathcal{M}$) is not overly restrictive for our purposes. Most importantly, a capacity class (i.e. a class of the form $\{m \in \mathcal{M} \mid m(A) \le v(A)$ for all Borel subsets, A, of U, $m(U) = v(U)\}$ for a capacity v) is weak*-compact ([6], Lemma 2.2) and convex. A capacity is a set function which generalizes the idea of a measure and was introduced by Choquet [12] and applied by Huber and Strassen [6] to model uncertainty in robust hypothesis testing. All the most commonly used "robustness neighborhoods" have been shown to be capacity classes (for the ϵ-mixture, total-variation, and Prokhorov models see [6] and [7]; for the band and p-point models see [8]). Thus these results are quite general.

Abstractly, the significance of <u>robust</u> signal estimation is clear: to be able to put the tightest upper bound on the error when the possibility of deviation from the assumed spectra exists is clearly desirable. However in most situations we must also expect that the robust estimator will not perform as well as the assumed (or nominal) estimator if the true spectra are the nominal spectra. So there is a trade-off. Thus the questions that naturally arise are how much is gained by the robust estimator in <u>its</u> worst case (at (m_S^L, m_N^L)) as compared to the nominal estimator at <u>its</u> worst case and how much is lost in using the robust estimator should the true spectra be the nominal ones. Clearly a blanket statement of the superiority of one estimator over the other in all cases is not likely to prove correct. Thus we undertake, in the next section, an attempt to answer these questions with some numerical examples.

3. AN EXAMPLE

In this section we consider robust filtering of a contaminated wide-sense Markov signal in white noise. Snyders [3] has shown that, if m_N^W represents white noise (i.e. $dm_N^W(\theta) = 2\pi N_0 d\theta$ for some positive constant N_0) and if $D(\theta)$ represents filtering (i.e. $D(\theta) \equiv 1$), then

$$e_D^+(dm_S, dm_N^W) = N_0\left\{1 - \exp\left[\frac{-1}{2\pi}\int_{-\pi}^{\pi} \log\left(1 + N_0^{-1}\frac{dm_S}{d\theta}(\theta)\right) \cdot d\theta\right]\right\} \quad (3.1)$$

for any signal spectrum m_S. We wish to consider the case when $\mathcal{N} = \{m_N^W\}$ and $\mathcal{S} = \{m \in \mathcal{M} \mid m(A) = (1-\epsilon)m_S^M(A) + \epsilon m'(A), \forall A \in \mathcal{A}$; for some nonnegative $m' \in \mathcal{M}$ with $m'(U) = m_S^M(U) = 2\pi v^2\}$ where m_S^M represents a nominal signal spectrum, here given by $dm_S^M(\theta) = [v^2(1-r^2)/(1-2r\cos\theta + r^2)]d\theta$ for $r \in [-1,1]$, and $\epsilon \in [0,1]$ represents the degree of possible contamination or error in assuming that m_S^M is the true signal spectrum.

The particular m_S^M we have chosen is the spectrum of a wide-sense Markov process. It has power v^2 and, for $r \ge 3 - 2\sqrt{3} \approx .172$, it has 3dB power bandwidth

$$\cos^{-1}[(r^2 - 4r + 1)/(-2r)] . \quad (3.2)$$

Substituting the expression for m_S^M into (3.1) we obtain $e_D^+(dm_S^M, dm_N^W)$. Alternatively, since $dm_S^M/d\theta$ is rational, we can determine that the optimal causal transfer function for the nominal pair (m_S^M, m_N^W) is given by (see [13], Chapter 7)

$$H_0^+(\theta) = \frac{K}{1 - \alpha e^{-i\theta}} \quad \theta \in U \quad (3.3)$$

where $\alpha = (b - \sqrt{b^2 - 4r^2})/2r$, $K = \alpha v^2(1-r^2)/(N_0(1-r\alpha))$

and $b = [v^2(1-r^2)/N_0 + (1+r^2)]$. We can then substitute (3.3), m_S^M and m_N^W into (2.2) to obtain $e_D^{+}(dm_S^M, dm_N^W)$. Further, this procedure may be used to obtain $e_D(dm_S, dm_N^W; H_0^{+})$ for any signal spectrum m_S; in particular, it may be used to obtain $e_D(dm_S^{WC}, dm_N^W; H_0^{+}) = \max_{m_S \in \mathscr{S}} e_D(dm_S, dm_N^W; H_0^{+})$. It is easy to see that m_S^{WC} (WC stands for worst case) is given by $(1-\epsilon)m_S^M + \epsilon m_S'$ where $m_S'(A) = 2\pi v^2$ iff $0 \in A$, since $|1 - H(0)|^2 = \sup_{\theta \in U} |1 - H(\theta)|^2$.

So we have error expressions for H_0 at the nominal pair and in its worst case. If there is a significant difference between these errors (i.e. if $e_D(dm_S^{WC}, dm_N^W; H_0^{+})$ is much larger than $e_D^{+}(dm_S^M, dm_N^W)$) for reasonable values of ϵ (say $\epsilon = .1$) then a clear need for robust filtering exists. The (minimax) robust filter H_R^{+} designed in Section 2, will satisfy this need if $e_D^{+}(dm_S^L, dm_N^W)$ is relatively close to $e_D^{+}(dm_S^M, dm_N^W)$. Later in this section we will show that this is often the case. First we must find the least-favorable signal spectrum, m_S^L, so that we may use (3.1) to calculate $e_D^{+}(dm_S^L, dm_N^W)$.

Because of the power constraint (i.e., $m(U) = v^2 \ \forall\, m \in \mathscr{S}$) we may use the results of Section III of [2] (particularly Lemma 1) and of [14] to say that

$$\frac{dm_S^L(\theta)}{d\theta} = \begin{cases} \dfrac{2\pi v^2(1-\epsilon)(1-r^2)}{1 - 2r\cos\theta + r^2} & \text{if } \dfrac{dm_S^M}{d\theta}(\theta) > 2\pi c \\[2ex] 2\pi v^2 \ (1-\epsilon)c & \text{if } \dfrac{dm_S^M}{d\theta}(\theta) \le 2\pi c \end{cases} \tag{3.4}$$

where c is a positive constant which can always be determined so that $\int_{-\pi}^{\pi} dm_S^L(\theta) = 2\pi v^2$. We may now substitute (3.4) into (3.1) to calculate $e_D^{+}(dm_S^L, dm_N^W)$, the worst case error for the robust filter H_R.

For comparison purposes the three error expressions discussed above ($e_D^{+}(dm_S^M, dm_N^W)$, $e_D(dm_S^{WC}, dm_N^W; H_0^{+})$ and $e_D^{+}(dm_S^L, dm_N^W)$) were normalized by the signal power v^2 and viewed as functions of the input signal-to-noise ratio (SNR) for each value of the uncertainty parameter ϵ and the bandwidth parameter r. (Recall that r completely determines the signal bandwidth via (3.2).) These three normalized error quantities were evaluated for a variety of values of the parameters ϵ and r and for a large range of input SNR (which for convenience was converted to dB).

Careful examination of FIG. 1 (here the 3dB power bandwidth is .001, $\epsilon = .1$ and $c = .121$) shows that for SNR's between 20 dB and 40 dB the worst case performance of the nominal filter ($e_D(dm_S^{WC}, dm_N^W; H_0^{+})$) is an order of magnitude or more greater than the worst case performance of the robust filter ($e_D^{+}(dm_S^L, dm_N^W)$). On the other hand at 20 dB this worst case performance of the robust filter is less than half an order of magnitude greater than that of the nominal filter at the nominal pair ($e_D^{+}(dm_S^M, dm_N^W)$, the "optimal" performance for the original (nonrobust) problem). Further this difference decreases rapidly, so that from 40 dB on there is essentially no difference between the <u>worst</u> performance of the robust filter and the performance of the nominal filter when the signal spectrum is as originally assumed; i.e., above 40 dB we lose nothing by using the robust filter and can at the same time gain a considerable improvement in worst case performance.

While the situation pictured in FIG. 1 is typical of most cases we encountered, FIG. 2 (here the bandwidth is .105, $\epsilon = .1$ and $c = .252$) is representative of some other cases. The differences among the three error quantities are small throughout the pictured range (and in fact they are smaller elsewhere). Here there is little need for robust filtering and of course the *robust filter accomplishes* little. It should be noted, however, that while the differences are small, we see that for input SNR close to 0 dB the worst case of the robust filter is as bad as that of the nominal filter. However, we have found throughout our investigation that the only situations in which the robust filter is likely to be disadvantageous are situations (such as in FIG. 2) when all the errors of both filters are too large to be practical.

4. CONCLUSIONS

In this paper we have shown that (minimax) robust causal smoothers, filters, and predictors can be sought by maximizing the functional $e_D^{+}(dm_S, dm_N)$ if the spectral classes $\mathscr{S}$ and $\mathscr{N}$ are convex and satisfy a topological condition which all commonly used models of uncertainty are known to satisfy. Further, for the case of a contaminated wide-sense Markov signal in white noise we have demonstrated, for a wide variety of input SNR's, that the trade-off of possible lessened nominal case performance in return for the improved worst-case performance of the robust filter is a very good one. Moreover, this particular case is a very reasonable model for many applications.

While the numerical results of Section 3 validate the analytical results of Section 2 for robust filtering, these analytical results are much broader. Thus, it would be interesting to

consider a numerical treatment of both the robust smoothing and prediction problems, analytical solutions to which may be obtained via the methods of Section 2.

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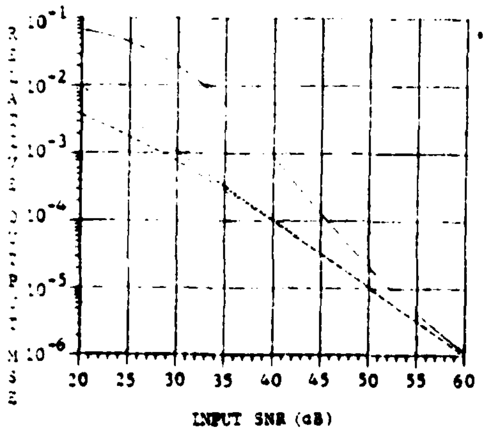

FIG. 1: Nominal filter at worst case (solid line), robust filter at worst case (dotted line), and nominal filter at nominal case (dashed line) when bandwidth = .001 and ϵ = .1.

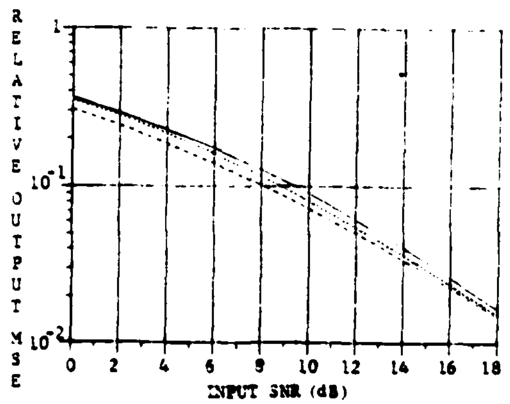

FIG. 2: Nominal filter at worst case (solid line), robust filter at worst case (dotted line), and nominal filter at nominal case (dashed line) when bandwidth = .105 and ϵ = .1.

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